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Reduced Differential Transform Method for Singularly Perturbed Sixth-Order Boussinesq Equation

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Abstract

The purpose of this paper is to obtain the approximate solution of singularly perturbed sixth-order Boussinesq equation and ill-posed Bouissnesq equation (for $\varepsilon = 0$) by the reduced differential transform method (RDTM). This numerical method for solving a wide variety of linear and nonlinear partial differential equations and usually gets the solution in a series form. The suggested algorithm is quite efficient and is practically well suited for use in these problems. Several examples are presented to demonstrate the efficiency and reliability (Because this method yield the desired accuracy only in a few terms and in a series form of the exact solution) of the RDTM, numerical results are discussed, compared with Exact solution. The numerical results show that this method is a powerful tool for solving nonlinear singular perturbed PDEs and the results show that the method reduces the numerical calculations.

Keywords: Singularly Perturbed Sixth-Order Boussinesq Equation; iIl-posed Boussinesq Equation; Reduced Differential Transform Method.

1. Introduction

We consider the following the ill-posed Boussinesq equation and sixth-order singularly perturbed Boussinesq equation, respectively:

$$u_{tt} = u_{xx} + u_{xx}^2 + u_{xxxx} \ , \ \varepsilon = 0 \ (1)$$

and

$$u_{tt} = u_{xx} + u_{xx}^2 + u_{xxxx} + \varepsilon u_{xxxxxx} \qquad . \quad (2)$$

The Boussinesq equation was modeled by Boussinesq in 1872 [12]. Darapi and Hua [1] introduced the singularly perturbed Boussinesq equation as a dispersive regularization of the ill-posed classical Boussinesq equation for $\varepsilon = 0$ in Equation (2), where $\varepsilon > 0$ is a small parameter.

This equation actually describes the bi-directional propagation of small amplitude and long capillary-gravity waves on the surface of shallow water for Bond number (surface tensionparameter) less than but very close to 1/3. The fourth-order Boussinesq equation has the traveling-solitary-wave solutions. In recent years, There are various techniques for solving singular perturbed sixth-order boussinesq equation, such that Qazi Mahmood Hassan and Syed Tauseef Mohyud-Din [9]. The ill-posed Boussinesq equation could be converted to an ordinary differential equation. Ying Wang [11] existence and nonexistence of solutions for a generalized Boussinesq equation established under certain conditions. Syed Tauseef Mohyud-Din and Muhammad Aslam Noor [10] solve singularly perturbed sixth-order Boussinesq equation using the convex homotopy method.

Daripa and Dash [2] comment on the physical relevance of singulary perturbed Boussinesq equation in the contex of water waves. Daripa and Dash [4] proved that the traveling wave solutions of Equation (2) are weakly nonlocal solitary waves characterized by small amplitude fast oscillations in the far-field and obtained weakly nonlocal solitary wave solutions of Equation (2). Z. Feng [3] investigated the generalized Boussinesq equation including the singularly perturbed Boussinesq equation

$$u_{tt} = [Q(u)]_{xx} + \sum_{i=1}^{n} b_i u_{(2i+2)x}$$

where $Q(u) = u + b_0 u^r, r$ and b_i are all real constants.If we choices $r = 2,\ n = 2, b_0 = 1, b_1 = 1$ and $b_2 = \varepsilon$, the singularly perturbed Boussinesq equation Equation (2) occurs. C. Song, H. Li, and J. Li [6] discussed the initial boundary value problem for the singularly perturbed Boussinesq-type equation

$$u_{tt} = u_{xx} + \sigma u_{xx} + \alpha u_{xxxx} + \beta u_{xxxxxx}\ , \qquad x \in (0,1), \qquad t > 0$$

$$u_x(0,t) = u_x(1,t) = u_{x^3(0,t)} = u_{x^5}(0,t) = u_{x^5}(1,t) = 0, \qquad t > 0$$

$$u(x,0) = u_0(x)\ , \qquad u_t(x,0) = u_1(x), \qquad x \in \overline{(0,1)}$$

where $\alpha > 0$ and $\beta > 0$ are real numbers, $u_0(x), u_1(x)$are given initial value functions. Changming Song, Jina Li and Ran Gao [8]. The nonexistence of global solution to the initial boundary value problem for the singularly perturbed Boussinesq-type equation is discussed.

In this paper, we applied the RDTM, which is the modified version of differential transform method (DTM), to ill-posed boussinesq and singularly perturbed sixth-order boussinesq equations. RDTM doesn't require any discretization or linearization. Also, it provides an analytical approximation, in many cases an exact solution, in a rapidly convergent sequence with elegantly computed terms. The partial differential equations have numerous essential applications in various fields of science and engineering such as fluid mechanic, thermo dynamic, heat transfer, physics [13]. These are discussed in Y. Keskin and G. Oturanc 2009 [5], J. K. Zhou 1986 [7], Hosseinzadeh, H. and Salehpour E. [13], P.K. Gupta 2011 [14], K. Yıldırım, B. İbiş and M. Bayram 2012 [15], A. Haghbin and S. Hesam 2012 [16], Y. Keskin and G. Oturanc 2010 [17], CL. Chen and Y.C. Liu 1998 [18], Benhammouda, B., Vazquez-Leal,H. and Sarmiento-Reyes, A. [19], F. Kangalgil and F. Ayaz 2009 [20], Y. Keskin and G. Oturanc 2010 [21], Mahmoud Rawashdeh 2013 [22].

2. Reduced Differential Transform Method (RDTM)

From the similar meaning of definition of Differential Transform Method and its properties, we can write the transforming form of RDTM [5,7,13 - 17,21 - 23].

If function $u(x,t)$is analytic and differentiated continuously with respect to t time and x space in the domain of interest, then let transformed function,

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} u(x,t)\right]_{t=0}. \qquad (3)$$

Thus, from Equation (3) it can be written the inverse transform of a sequence $\{U_k(x)\}_{k=0}^{\infty}$ as:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x) t^k = U_0(x) + U_1(x) + U_2(x)t^2 + U_3(x)t^3 + \cdots \qquad (4)$$

where the t- dimensional spectrum function $U_k(x)$ is the transformed function. In this paper, the lowercase $u(x,t)$ represent the original function while the uppercase $U_k(x)$ stand for the transformed function.

If we consider the expressions Equation (3) and (4), it can be found that the concept of the RDTM is derived from the power series expansion. The fundamental mathematical operations performed by RDTM can be readily obtained and are listed in Table 1.

Table 1: The transformation of RDTM for some functions.

Functional Form	**Transformed Form**
$u(x,t)$	$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0}$
$w(x,t) = u(x,t) \pm v(x,t)$	$W_k(x) = U_k(x) \pm V_k(x)$
$w(x,t) = \alpha u(x,t)$, α is a constant.	$W_k(x) = \alpha U_k(x)$
$w(x,t) = u(x,t)v(x,t)$	$W_k(x) = \sum_{r=0}^{k} U_r(x)V_{k-r}(x)$
$w(x,t) = \frac{\partial^2 u(x,t)}{\partial t^2}$	$W_k(x) = (k+1)(k+2)\, U_{k+2}(x)$
$w(x,t) = \frac{\partial^r u(x,t)}{\partial x^r}, r = 0,1,2,3 \ldots$	$W_k(x) = \frac{\partial^r u(x,t)}{\partial x^r}$

3. Numerical Examples

In this section, we solve singularly perturbed sixth-order Boussinesq equation and Il-posed Boussinesq equation by applying RDTM. The results have been provided by MAPLE.

Example 3.1

First,we consider the singularly perturbed sixth-order Boussinesq equation for $0 < \varepsilon = \frac{1}{2} << 1$ [10]

$$u_{tt} = u_{xx} + u_{xx}^2 - u_{xxxx} + \frac{1}{2}u_{xxxxxx} \qquad (5)$$

with initial condition

$$u(x,0) = \frac{-105}{169} sech^4\left(\frac{x}{\sqrt{26}}\right), \qquad u_t(x,0) = \frac{-210}{2197}\sqrt{\frac{194}{13}}\, sech^4\left(\frac{x}{\sqrt{26}}\right) tanh\left(\frac{x}{\sqrt{26}}\right) (6)$$

$$u(x,t) = -\frac{105}{169} sech^4[\frac{1}{\sqrt{26}}\left(x - \sqrt{\frac{97t}{169}}\right)]$$

is the exact solution.

We get according to Equation (4), we obtain from the initial conditions as:

$$U_0(x) = -\frac{105}{169}\,\mathrm{sech}(\frac{1}{26}x\sqrt{26})^4$$

$$U_1(x) = -\frac{210}{28561}\sqrt{2522}\,\mathrm{sech}(\frac{1}{26}x\sqrt{26})^4\,\mathrm{tanh}(\frac{1}{26}x\sqrt{26})$$

Using the RDTM and according to Table 1, we obtain recursive formula for k=0,1,2,3… from Equation (5) as:

$$U_{k+2}(x) = \frac{1}{(k+1)(k+2)}\left[\frac{\partial^2}{\partial x^2}U_k(x) + 2\left(\frac{\partial}{\partial x}U_k(x)\right)^2 + 2U_k(x)\left(\frac{\partial^2}{\partial x^2}U_k(x)\right) - \left(\frac{\partial^4}{\partial x^4}U_k(x)\right) + \frac{1}{2}\frac{\partial^6}{\partial x^6}U_k(x)\right]$$

$$k = 0, U_2(x) = -0.1911697770\,\mathrm{sech}(0.1961161351x)^4\,\mathrm{tanh}(0.1961161351x)^2 +$$

$$+0.04779244424\operatorname{sech}(0.1961161351x)^4+\cdots$$

Thus, we obtain:

$$u_{RDTM}(x,t)=U_0(x)+U_1(x)t+U_2(x)t^2+\cdots=$$

$$-\frac{105}{169}\operatorname{sech}\left(\frac{1}{26}x\sqrt{26}\right)^4+\left(-\frac{210}{28561}\sqrt{2522}\operatorname{sech}\left(\frac{1}{26}x\sqrt{26}\right)^4\tanh\left(\frac{1}{26}x\sqrt{26}\right)\right)t+$$

$$(-0.1911697770\operatorname{sech}(0.1961161351x)^4\tanh(0.1961161351x)^2++0.04779244424\operatorname{sech}(0.1961161351x)^4+\cdots)t^2+\cdots$$

Consequently, RDTM solution converges to the exact solution of the singularly perturbed sixth-order Boussinesq equation.

Using our method, we choose 7 points on [-1,1] respectively.The numerical results are given in the following Table 2 and Table 3.

Table 2: Comparison of the RDTM solution with EXACT solution for $t=0.01$

x	y_{Exact}	y_{RDTM}	Error
-1	-0.5687548626	-0.5749176725	0.0061628100
-0.4	-0.5999175147	-0.6039478192	0.0040303034
-0.8	-0.6176798236	-0.6192464778	0.0015666540
0	-0.6210275342	-0.6210321299	0.0000045957
0.4	-0.6163009110	-0.6139924936	0.0023084170
0.8	-0.5968130609	-0.5921188393	0.0046942220
1	-0.5819965184	-0.5762424655	0.0057540530

Table 3: Comparison of the RDTM solution with EXACT solution for $t=0.001$

x	y_{Exact}	y_{RDTM}	Error
-1	-0.5734669034	-0.5755158733	0.0020489700
-0.4	-0.6030158197	-0.6043289214	0.0013131020
-0.8	-0.6189100360	-0.6193790454	0.0004690094
0	-0.6212743446	-0.6213017477	0.0000073017
0.4	-0.6145862196	-0.6137381107	0.0008481089
0.8	-0.5932811273	-0.5916287013	0.0016524260
1	-0.5776560680	-0.5756483525	0.0020077160

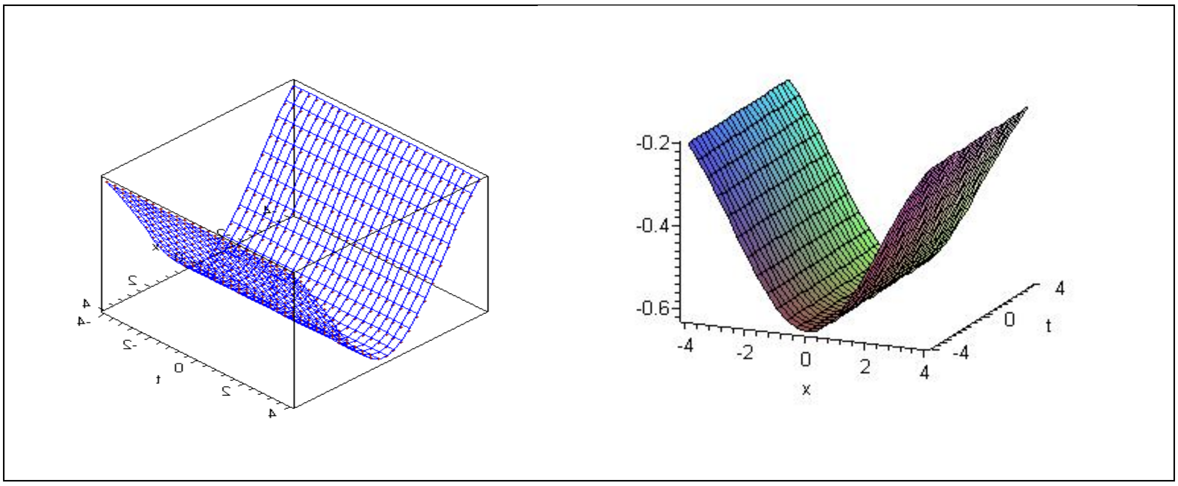

Figure 1: First figüre is comparison of the RDTM solution and the EXACT solution, second figure is $u(x,t)_{RDTM}$ the solution of Example 1.

Example 3. 2

Now, we consider Il-posed Boussinesq equation for $\varepsilon = 0$ [10]

$$u_{tt} = u_{xx} + 3u_{xx}^2 - u_{xxxx} \qquad (7)$$

with the initial conditions ,

$$u(x,0) = \frac{2ak^2 e^{kx}}{(1+ae^{kx})^2}$$

$$u_t(x,0) = \frac{2as^3\sqrt{1+s^2}(1-ae^{sx})e^{sx}}{(1+ae^{sx})^3}$$

where a,s arbitrary constants. The exact solution,

$$u(x,t) = \frac{2as^2\exp(sx + s\sqrt{1+s^2}t)}{\left(1 + a\exp\left(sx + s\sqrt{1+s^2}t\right)\right)^2}.$$

Using the RDTM, we get according to Equation (3), we obtain recursive formula for k=0,1,2,3,… and from the initial conditions,

$$U_{k+2}(x) = \frac{1}{(k+1)(k+2)}\left[\frac{\partial^2}{\partial x^2}U_k(x) + 3\left(\frac{\partial}{\partial x}U_k(x)\right)^2 + 3U_k(x)\left(\frac{\partial^2}{\partial x^2}U_k(x)\right) - \left(\frac{\partial^4}{\partial x^4}U_k(x)\right)\right]$$

$$U_0(x,t) = \frac{2e^x}{(1+e^x)}$$

$$U_1(x,t) = \frac{2\sqrt{2}(1-e^x)e^x}{(1+e^x)^3}$$

$$U_2(x,t) = \frac{2e^x}{(1+e^x)^2} - \frac{36e^{2x}}{(1+e^x)^3} + \frac{156e^{3x} + 24e^{2x}}{(1+e^x)^4} - \frac{120e^{3x} - 240e^{4x}}{(1+e^x)^5} + \frac{120e^{4x} + 120e^{5x}}{(1+e^x)^6}$$

$$u(x,t)_{RDTM} = \frac{2e^x}{(1+e^x)^2} + \frac{2.828427124(1-e^x)e^x t}{(1+e^x)^3} + \left(\frac{2e^x}{(1+e^x)^2} - \frac{36e^{2x}}{(1+e^x)^3} + \frac{156e^{3x}}{(1+e^x)^4} + \frac{24e^{2x}}{(1+e^x)^4} - \frac{120e^{3x}}{(1+e^x)^5} + \frac{120e^{4x}}{(1+e^x)^6} - \frac{240e^{4x}}{(1+e^x)^5} + \frac{120e^{5x}}{(1+e^x)^6}\right)t^2.$$

Thus, RDTM solution converges to the exact solution of the ill-posed singularly perturbed sixth-order Boussinesq equation. Using our method, we choose 7 points on [-1,1] for$a = 1, s = 1$. The numerical results are given in the following Table 4 and Table 5.

Table 4: Comparison of the RDTM solution with EXACT solution for $t = 0.01$

x	y_{Exact}	y_{RDTM}	Error
-1	0.4999968906	0.3957866466	0.10427731
-0.4	0.4800458204	0.4301060637	0.02013953
-0.8	0.4270071640	0.4957068896	0.01999250
0	0.3923166756	0.4999750000	-0.10943970
0.4	0.3164133330	0.4791589895	-0.16539060
0.8	0.2425897230	0.4255084686	-0.18608590
1	0.2091905022	0.3906469564	-0.18471300

Table 5: Comparison of the RDTM solution with EXACT solution for $t = 0.001$

x	y_{Exact}	y_{RDTM}	Error
-1	0.4977580882	0.3934807804	0.10427731
-0.4	0.4778963516	0.4577568187	0.02013953
-0.8	0.4250951830	0.4051026810	0.01999250
0	0.3905600256	0.4999997500	-0.10943970
0.4	0.3149965504	0.4803871508	-0.16539060
0.8	0.2415034954	0.4275893919	-0.18608590
1	0.2082538240	0.3929668115	-0.18471300

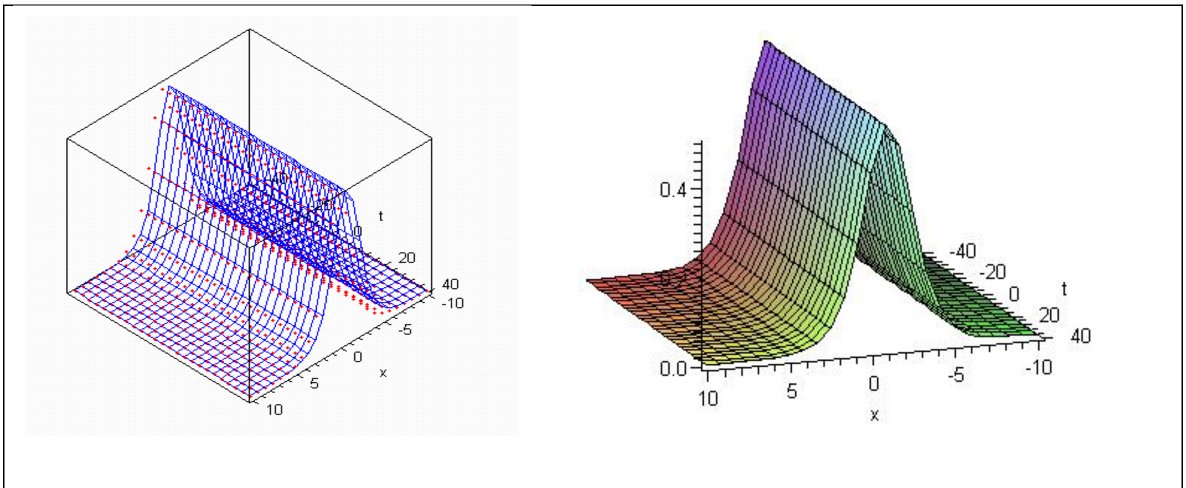

Figure 2: First figüre is comparison of the RDTM solution and the EXACT solution, second figure is $u(x,t)_{RDTM}$ the solution of Example 2.

4. Conclusion

We introduced reduced differential transform method (RDTM) for solving singularly perturbed sixth-order Boussinesq and ill-posed Boussinesq Equations. The main advantage of the RDTM is to provide the user an analytical approximation to the solution, in many cases, an exact solution, in a rapidly convergent sequence with elegantly computed terms. The solution procedure of the RDTM is simpler and effective. The results show that the RDTM is a powerful and effectiveness method for solving singularly perturbed equations

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