Mathematics and Statistics: Open Access
Received: Jan 27, 2016, Accepted: Apr 02, 2016, Published: Apr 06, 2016
Math Stat, Volume 2, Issue 2
http://crescopublications.org/pdf/msoa/MSOA-2-013.pdf
Article Number: MSOA-2-013

Research Article Open Access

The Condition of the Fourth-Order Finite Element Global Stiffness Matrix Produced via the Principle of Minimum Potential Energy and via Least-Squares

Isaac Fried* and Yufei Zhang

Department of Mathematics, Boston University, Boston, MA 02215, USA

***Corresponding Author**: Isaac Fried, Department of Mathematics, Boston University, Boston, MA 02215, USA, E-mail: if@math.bu.edu

Citation: Isaac Fried and Yufei Zhang (2016) The condition of the fourth-order finite element global stiffness matrix produced via the principle of minimum potential energy and via least-squares. Math Stat 2: 013.

1. Introduction

Actual, a-priory bounds [1-3], depending on the discretization parameters, are obtained for the spectral condition number of the global stiffness matrices generated by finite elements (FE) from a variationally based discretization, as well as from a least-squares (LS) based discretization [4, 5, 7] of the fourth order problem. It is shown that the spectral condition number of the global stiffness matrix is $O(h^{-4})$ for the variationally derived matrices, but only $O(h^{-2})$ for the LS derived matrices of the fourth-order boundary value problems, first reduced to a coupled system of first order equations. See also [6], Sec. 6, and [4] Sec. 2.7.4.

Also considered here is the LS imposition of all boundary conditions, and the optimal weight choice.

2. Variational Principle of Minimum Potential Energy

Consider the fourth-order, beam, BVP

$y^{(4)} = f(x),\ 0 \le x \le 1,\ y(0) = y''(0) = 0,\ y'(1) = y'''(1) = 0$ (1)

and let **y** be a trial function approximating solution y of eq.(1), and such that

$$\tilde{y} \in C^1[0,1], \tilde{y}(0) = \tilde{y}'(1) = 0. \quad (2)$$

For this space of continuous and having a continuous derivative trial functions satisfying but the essential boundary conditions, we derive, by expansion and some integrations by parts, the variational principle

$$\frac{1}{2}\int_0^1 (y'' - \tilde{y}'')^2 dx = \pi(\tilde{y}) - \pi(y) \ge 0 \quad (3)$$

with $\pi(\tilde{y}) - \pi(y) = 0$ only if $y = \tilde{y}$, and where the total potential energy functional is

$$\pi(\tilde{y}) = \int_0^1 \left(\frac{1}{2}\tilde{y}''^2 - f\,\tilde{y}\right)dx. \quad (4)$$

The total potential energy in eq. (3) is formed by finite elements, then upon minimization

$$\nabla \pi(\tilde{y}) = 0 \rightarrow Ky = F \quad (5)$$

is reduced to a system of linear equations. We will be interested in the condition of the stiffness matrix K of the global system of equations.

3. Boundedness of the Linear Fourth Order Differential Operator

We determine the minimum value λ of

$$R[y]=\frac{\int_0^1 y''^2\,dx}{\int_0^1 y^2 dx},\ y\in C^2[0,1],\ y(0)=y'(1)=0$$

(6)

we setup for *R[y]* the Euler-Lagrange equation by replacing the minimizing *y* by $y+\in\eta, \eta(0)=\eta'(1)=0$, and obtain by the fundamental theorem of the calculus of variations [8] from

$$\left(\frac{\partial R}{\partial \in}\right)_{\in=0}=0 \qquad (7)$$

after some integration by parts, the characteristic differential equation

$$y^{(4)}-\lambda y=0,\ y(0)=y''(0)=0,\ y'(1)=y'''(1)=0$$

(8)

solved for the minimal λ by

$$y=\sin\frac{\pi}{2}x,\ \lambda=\left(\frac{\pi}{2}\right)^4 \qquad (9)$$

and

$$R[y]\ge\lambda\succ 0 \qquad (10)$$

for any, admissible, $y\in C^1[0,1], y(0)=y'(1)=0$.

4. General, A-Priori, Bounds on the Extremal Eigenvalues of the Global FE Mass and Stiffness Matrices

Element stiffness matrix *k* and global stiffness matrix *K* hold the coefficients of the quadratic part of the total potential energy $\tilde{\pi}=\pi(\tilde{y})$. For $\tilde{\pi}(\tilde{y})$ in eq. (4) they are

$$y^TKy=\int_0^1 \tilde{y}''dx,\ y_e^Tky_e=\int_{x_1}^{x_2}\tilde{y}''^2\,dx,\ x_2-x_1=h,\ \tilde{y}\in C^1 \qquad (11)$$

where y_e is the element vector of nodal values for the polynomial approximant of y inside the typical finite element. Vector *y* is the global vector of nodal values for the entire mesh.

Element stiffness matrix *k* is symmetric and only positive semi-definite.

Similarly, we define element mass matrix *m* and global mass matrix *M* as

$$y^TMy=\int_0^1\tilde{y}^2dx,\ u_e^Tmu_e=\int_{x_1}^{x_2}\tilde{y}^2dx,\ x_2-x_1=h \qquad (12)$$

The element shape functions are linearly independent and, consequently, element mass matrix *m* is symmetric and positive definite, λ (m) > 0.

Assembly of the element quadratic forms over the entire mesh creates global stiffness matrix *K*, and global mass matrix *M* of the entire system of elements.

The boundedness of the linear differential operator, as in eq. (10), implies that

$$\frac{y^TKy}{y^TMy}\ge\lambda\succ 0 \ \text{ or }\ y^TKy\ge\lambda y^TMy \qquad (13)$$

for any global vector *y* that satisfies the end conditions. Let vector *y* be the normalized, y^Ty=1, eigenvector corresponding to λ_1 (K), the lowest eigenvalue of global matrix *K*. Then

$$\lambda_1(K)\ge\lambda\lambda_1(M) \qquad (14)$$

since $y^TMy \ge \lambda_1$(M) for any normalized vector y.

Summation of all quadratic forms over the entire finite elements mesh forms global mass matrix M

$$y^T My = \sum_{e=1}^{e=Ne} y_e^T m y_e \quad (15)$$

and for each element

$$0 \prec \lambda_1(m) \le \frac{y_e^T m y_e}{y_e^T y_e} \le \lambda_n(m) \qquad (16)$$

where λ_1(m) is the lowest eigenvalue, and λ_n(m) is the highest eigenvalue of the positive definite and symmetric element mass matrix m. Hence

$$\lambda_1(m)\sum_{e=1}^{e=Ne} y_e^T y_e \le y^T My \le \lambda_n(m)\sum_{e=1}^{e=Ne} y_e^T y_e \qquad (17)$$

or, since $y^T y$ = 1, and since in the one dimensional case, no more than two elements share a common nodal value,

$$\lambda_1(m) \le \lambda(M) \le 2\lambda_n(m). \qquad (18)$$

In other words: the least eigenvalue of global matrix M is superior to the the least eigenvalue of element matrix m.

For the global stiffness matrix K, the last inequalities are

$$0 = \lambda_1(k) \le \lambda(K) \le 2\lambda_n(k) \qquad (19)$$

with a trivial lower bound due to the fact that element stiffness k is only positive semidefinite.

Combining eqs. (14), (18) and (19) we obtain the bound on the spectral condition number of global stiffness matrix K

$$C_2(K) = \frac{\lambda_n(K)}{\lambda_1(K)} \le \frac{2}{\lambda}\frac{\lambda_n(k)}{\lambda_1(m)} \qquad (20)$$

writtent in terms of characteristic value λ of the differential operator in question, the smallest eigenvalue λ_1(m) of element mass matrix m, and the largest eigenvalue λ_n(k) of element stiffness matrix k.

4.1 Cubic Beam Finite Element

An admissible approximant for $\pi(\tilde{y})$ of eq. (4), in each element is of the C^1, Hermite, function

$$\tilde{y} = y_1\phi_1 + y_1'\ \phi_2 + y_2\phi_3 + y_2'\ \phi_4 \qquad (21)$$

for which the cubic shape functions are

$$\phi_1 = 1 - 3\xi^2 + 2\xi^3, \phi_2 = h(\xi - 2\xi^2 + \xi^3), \phi_3 = 3\xi^2 - 2\xi^3, \phi_4 = h(-\xi^2 + \xi^3) \qquad (22)$$

$$x = x_1 + h\xi, dx = hd\xi, h = x_2 - x_1, 0 \le \xi \le 1. \qquad (23)$$

We write

$$\tilde{y} = y_e^T\phi = \phi^T y_e \text{ and } \tilde{y}' = y_e^T\phi' = \phi'^T\ y_e \qquad (24)$$

Where

$$y_e = \begin{bmatrix} y_1 \\ y_1' \\ y_2 \\ y_2' \end{bmatrix}, \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix} \qquad (25)$$

and have that

$$y_e^T m y_e = \int_{x_1}^{x_2} y^2 dx = y_e^T [\int_{x_1}^{x_2} \phi\phi^T dx] y_e \text{ and } y_e^T k y_e = \int_{x_1}^{x_2} y'^2 dx = y_e^T [\int_{x_1}^{x_2} \phi' \phi'^T dx] y_e. \quad (26)$$

from which we obtain the element mass matrix

$$m = h\int_0^1 \phi_i \phi_j d\xi = \frac{h}{420}\begin{bmatrix} 156 & 22h & 54 & -13h \\ 22h & 4h^2 & 13h & -3h^2 \\ 54 & 13h & 156 & -22h \\ -13h & -3h^2 & -22h & 4h^2 \end{bmatrix} \quad (27)$$

And the element stiffness matrix

$$k = \frac{1}{h^3}\int_0^1 \phi_i'' \phi_j'' d\xi = \frac{1}{h^3}\begin{bmatrix} 12 & 6h & -12 & 6h \\ 6h & 4h^2 & -6h & 2h^2 \\ -12 & -6h & 12 & -6h \\ 6h & 2h^2 & -6h & 4h^2 \end{bmatrix}. \quad (28)$$

Matrix m is symmetric and positive definite. Matrix k is symmetric and positive semidefinite being of only rank 2. In fact, the four eigenvalues of element stiffness matrix k and element mass matrix m (scaled for $h = 1$ inside the element) are

$$\lambda_1(k) = 0, \lambda_2(k) = 0, \lambda_3(k) = 2h^{-3}, \lambda_4(k) = 30h^{-3} \quad (29)$$

And

$\lambda_1(m) = 0.000486h$, $\lambda_2(m) = 0.00270h$, $\lambda_3(m) = 0.245h$, $\lambda_4(m) = 0.514h$.

Now we have directly from eq. (21) that

$$C_2(M) = 0(1), C_2(K) = 0(h^{-4}). \quad (30)$$

For a cantilever beam (y(0) = 0, y'(1) = 0, y''(1) = 0,y'''(1) = 0) discretized by cubic elements (scaled so that $h = 1$ inside the matrix) we actually compute for Ne = {1, 2, 3,4,5} the condition numbers $C_2(K)$ = {19,184, 668,1750, 3801}, represented by the formula

$$C2(K) = 14(Ne)^{3.48}. \quad (31)$$

4.2 Least-Squares Fourth-Order Problem

We factor the fourth-order equation

$$y^{(4)} = f(x), 0 \le x \le 1. \quad (32)$$

into the chain of first-order equations

$$y' = u, \ u' = v, \ v' = w, \ w' = f \quad (33)$$

then reformulate it as the least-squares solution minimizing the quadratic error functional

$$E(y,u,v,w)=\int_0^1((y'-u)^2+(u'-v)^2+(v'-w)^2+(w'-f)^2)dx \quad (34)$$

ver y,u,v,w satisfying their end values.

5. **Boundedness of the LS Differential Operator**

Minimizing the quotient

$$R(y,u,v,w)=\frac{\int_0^1((y'-u)^2+(u'-v)^2+(v'-w)^2+w'^2)dx}{\int_0^1(y^2+u^2+v^2+w^2)dx}\geq\lambda\succ 0 \quad (35)$$

under the boundary conditions

$$y(0) = 0,\ u(1) = 0,\ v(0) = 0,\ w(1) = 0 \quad (36)$$

we obtain from the fundamental theorem of the calculus of variations the following system of Euler-Lagrange equations

$$\begin{aligned} &y''+\lambda y=u',\ y(0)=0,\ y'(1)=0\\ &u''-u+\lambda u=-y'+v',\ u(1)=0,\ u'(0)=0\\ &v''-v+\lambda v=-u'+w',\ v(0)=0,\ v'(1)=0\\ &w''-w+\lambda w=-v',\ w(1)=0,\ w'(0)=0 \end{aligned} \quad (37)$$

satisfied by

$$y=\sin\frac{\pi}{2}x,\ u=A(\lambda)\cos\frac{\pi}{2}x,\ v=B(\lambda)\sin\frac{\pi}{2}x,\ w=C(\lambda)\cos\frac{\pi}{2}x \quad (38)$$

with the charcteristic equation for λ

$$\lambda^4 - 12.8696\lambda^3 + 54.3329\lambda^2 - 84.2859\lambda + 37.0646 = 0 \quad (39)$$

furnishing the minimal $\lambda = 0.72041$.

Using the finite elements of the the next section to minimize R of eq. (35) we obtain for the number of elements $Ne = \{8,12,16\}$ the values $\lambda(Ne) = (0.72837, 0.72394, 0.72239)$, from which we deduce that $\lambda(Ne) = \lambda + cNe^{-2}$. (40)

Elimination of constant c leaves us with the computed, extrapolated to the limit, or ultimate, $\lambda = 0.72041$, agreeing exactly with the theoretical value obtained from characteristic eq. (39).

Doing the same for the cantilever beam satisfying the end conditions $y(0) = u(0) = v(1) = w(1) = 0$, we similarly obtain the minimal $\lambda = 0.939781$.

6. Bounds on the Condition of the Global LS Matrices of the Fourth Order Problem

Element stiffness matrix k contains the coefficients of the approximated quadratic form part of E in eq. (34)

$$y_e^T k y_e = \int_e ((y'^2 + u'^2 + v'^2 + w'^2) + (u^2 + v^2 + w^2) - 2(y'u + u'v + v'w))dx. \quad (41)$$

A linear interpolations for y,u,v,w over an element of size h produces the element matrices

$$k_1 = \frac{1}{h}\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$k_2 = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}, k_3 = \frac{h}{6}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 \end{bmatrix} \quad (42)$$

and element stiffness $k = k_1 + k_2 + k_3$.

The element mass matrix m *is* here

$$y_e^T m y_e = \int_e (y^2 + u^2 + v^2 + w^2)dx, m = \frac{h}{6}\begin{bmatrix} 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 \end{bmatrix} \quad (43)$$

of the eigen values

$$\lambda_1(m)=\lambda_2(m)=\lambda_3(m)=\lambda_4(m)=\frac{h}{6},\lambda_5(m)=\lambda_6(m)=\lambda_7(m)=\lambda_8(m)=\frac{h}{2}. \quad (44)$$

If h is small,

$$\lambda_1(k)=\lambda_2(k)=0,\lambda_3(k)=\lambda_4(k)=\frac{h^3}{24},\lambda_5(k)=\lambda_6(k)=\lambda_7(k)=\lambda_8(k)=\frac{2}{h}, \quad (45)$$

And by eq. (20) the upper bound on the spectral condition number of the present global stiffness matrix K is

$$C_2(K)\le\frac{8}{\lambda}h^{-2}$$

(46)

in which λ is the minimum value of the least squares Rayleigh's functionals quotient.

By actual calculations we obtain for this global stiffness matrix K, under the end conditions $y(0)=y'(0)=y''(1)=y'''(1)=0,$ or $y(0)=u(0)=v(1)=w(1)=0$, of the cantilever beam, and for the number of elements Ne = {2, 4, 8, 16, 18}, the corresponding condition numbers

$$C_2(K)=\frac{\lambda_n(K)}{\lambda_1(K)}=\{20,80,304,1162,1461\} \quad (47)$$

That fit well the formula

$$C_2(K)=4.5h^{-2},h=\frac{1}{Ne}.$$

(48)

6.1 Least-squares Imposition of the Boundary Conditions

The weighted least-squares imposition of the boundary conditions y(0) = u(0) = 0, and u(1) = w(1) = 0 in the above fourth-order problem entails merely adding positive weight w to the diagonal entries $K(1,1)$, $K(2,2)$, $K(n-1,n-1)$, and $K(n,n)$ of global stiffness matrix K.

For a mesh of 8 elements, Ne = 8, and w = {1, 10, 20, 30}, we compute $C_2(K)$ = {1060, 380, 369, 437}, with a minimum of 364 occurring at w = 16. For a mesh of 16 elements, {4060, 1440, 1301, 1260, 1298}, with a minimum 1260 at w = 30, and an optimal $w = 2h^{-1} = 2Ne$.

Taking this optimal w, we compute for Ne = {4, 8, 16, 24, 36} the condition numbers $C_2(K)$ = {115, 360,1258, 2701, 5887}, which fit the formula $C_2(K) = 6.5\ Ne^{1.9}$.

7. Accuracy and Enforcement of Boundary Conditions

Solving the 4th order BVP $y'''' - 1 = 0$, $0 \le x \le 1$, $y(0) = y'(0) = y''(1) = y'''(1) = 0$, over six finite elements we obtain the approximation y_7 = 0.1239 for y(1) = 1/8 = 0.125. Also u_7 = 1.674 for y'(1) = 1/6 = 0.1667. Also v_1 = 0.49685 for y''(0) = 0.5. Also w_1 = - 0.9968 for y'''(0) = - 1.

Entering the boundary conditions via least squares, as in eq. (49) below, with weight 10 for the higher order boundary conditions for extra accuracy,

$$E(y,u,v,w)=\int_0^1((y'-u)^2+(u'-v)^2+(v'-w)^2+(w'-f)^2)dx+E_b,$$
$$E_b=y(0)^2+y'(0)^2+10y''(1)^2+10y'''(1)^2, \quad (49)$$

we obtain the approximation y_7 = 0.1238 for y(1) = 1/8 = 0.125. Also u_7 = 0.1673 for y'(1) = 1/6 = 0.1667. Also v_1 = 0.49667 for y"(0) = 0.5. Also wi = -0.99667 for y''' (0) = -1.

For the breakup of the two dimensional biharmonic equation in preparation for a least- squares solution see [8].

References

1. Fried I. Condition of finite element matrices generated from non-uniform meshes. AIAA Journal 1972; 10: 219-221.
2. Fried I. Bounds on the spectral and maximum norms of the finite element stiffness, flexibility and mass matrices. Int. J. Solids and Structures 1973; 9: 1013-1034.
3. Fried I. The condition of the least-squares finite element matrices. International Journal for Numerical Methods in Engineering Nov. 2015; DOI: 10.1002/nme.5146.
4. Jiang Bo-nan Least-Squares Finite Element Method. Springer: Berlin, 1998.
5. Bochev PB, Gunzburger MD. Least-Squares Finite Element Methods. Springer: Berlin, 2009.
6. Cai Z., Lazrov R., Manteuffel TA., and McCormick SF. First-order system least squares for second-order partial differential equations: part I. Siam J. Numer. Anal. 1994; 31, 6: 1785-1799.
7. Eason ED. A review of least-squares methods for solving partial differential equations. International Journal for Numerical Methods in Engineering 1976; 10, 5: 1021-1046.
8. Weinstock R. Calculus of Variations with Applications to Physics and Engineering. Dover: New-York, 1974.
9. Thatcher RW. A Least Squares Method for Solving Biharmonic Problems. SIAM Journal on Numerical Analysis, Vol. 38, No. 5; 2001: 1523-1539.