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Numerical Solution of the Coupled Viscous Burgers' Equation via Cubic Trigonometric B-spline Approach

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Abstract

This paper presents a new approach and methodology to solve one dimensional coupled viscous Burgers' equation with Dirichlet boundary conditions using cubic trigonometric B-spline collocation method. The usual finite difference scheme is applied to discretize the time derivative. Cubic Trigonometric B-spline basis functions are used as an interpolating function in the space dimension. The scheme is shown to be unconditionally stable using the von Neumann (Fourier) method. Two test problems are presented to confirm the accuracy of the new scheme and to show the performance of trigonometric basis functions. The numerical results are found to be in good agreement with known exact solutions and also with earlier studies.

Keywords: One dimensional coupled viscous Burgers' equation; Cubic trigonometric B-spline basis functions; Cubic trigonometric B-spline collocation method; Stability.

Introduction

In this paper, we are discussing the numerical solutions of one dimensional coupled Burgers' equations, proposed by Esipov [1]. This system of coupled Burgers' equation is a simple model of sedimentation or evolution of scaled volume concentrations of two kinds of particles in fluid suspensions or colloids under the effect of gravity [2]. The coupled Burgers' equation is given by

$$u_t + \alpha u u_x + \beta (uv)_x - u_{xx} = 0 \qquad x \in [a,b],\ 0 \le t \le T$$
$$v_t + \alpha v v_x + \eta (uv)_x - v_{xx} = 0 \qquad x \in [a,b],\ 0 \le t \le T \qquad (1)$$

with initial conditions

$$u(x,0) = u_0(x),\ v(x,0) = v_0(x) \qquad (2)$$

and boundary conditions

$$\begin{cases} u(a,t) = f_1(t) \quad , \ u(b,t) = f_2(t) \\ v(a,t) = g_1(t) \quad , \ v(b,t) = g_2(t) \end{cases} \qquad (3)$$

Where α, β and η are constant, and subscripts x and t denote differentiation of distance and time respectively. The coupled Burgers equations belong to an important class of basic flow equations [3], Ersoy and Idris [4] solved nonlinear coupled Burger Equation by Exponential Cubic B-spline Finite Element. Kutluay and Ucar [5]solved coupled Burgers' equation by the Galerkin quadratic B-splinefinite element method. Vineet et al.[6] used the fully implicit Finite-difference to solve one dimensional Coupled Nonlinear Burgers' equations.Kaya solved the coupled viscous burgers Equation by the decomposition method [7].Mittal and Tripathiused the Collocation Method for solved CoupledBurgers' Equations [8]. Mittal and Arora solved the coupled viscous Burgers' equations using cubic B-spline collocation scheme on the uniform mesh points based on Crank–Nicolsonformulation for time integration and cubic B-spline functions for space integration [9]. Mittal et al.Haar wavelet-based numerical investigation of coupled viscous Burgers' equation [10].Ghotbi et al.employed the homotopy perturbation method [11]. Rashid and Ismail used the Fourier pseudo-spectral method for finding the approximate solutions of the coupled Burgers' equation [12]. Abazari and Borhanifarobtained both numerical and analytical solutions of the Burgers' and coupled Burgers' equations using the differential transformation method [13]. Zhang et al.have extended the local discontinuous Galerkin method to solve Burgers' and coupled Burgers' equations [14].Siraj-ul-Islam et al.solved coupled Burgers' equationnumerically by a simple classical RBFs collocation (Kansa) method without using a mesh to discretize the problem domain [15]. Khater [20] and Rashid [21] solved the Burgers-type equations using a Chebyshev spectral collocation method and Chebyshev–Legendre Pseudo-Spectral method respectively.

In our work, a numerical collocation finite difference technique based on cubic trigonometric B-spline is presented for the solution of coupled viscous Burgers' equation (1) with initial conditions in equation (2) and boundary conditions in equations (3). A usual finite difference scheme is applied to discretize the time derivative while cubic trigonometric B-spline is utilized as an interpolating function in the space dimension. The proposed method is unconditionally stable and this is proved by von Neumann approach.

The outline of this paper is as follows: In section 2, cubic trigonometric B-spline scheme is explained. In section3, the method is described and applied to the coupled viscous Burgers' equation. In section 4, stability of the method is discussed. In section 5, numerical examples are included to establish the applicability and accuracy of the proposed method computationally. Conclusion is given in section 6.

Cubic Trigonometric B-Spline Functions

In this section, we define the cubic trigonometric basis function as follows [7, 8].

$$T_j^4(x) = \frac{1}{z}\begin{cases} q^3(x_j), & x \in [x_j, x_{j+1}) \\ q(x_j)\left(q(x_j)p(x_{j+2}) + p(x_{j+3})q(x_{j+1})\right) + p(x_{j+4})p^2(x_{j+1}), & x \in [x_{j+1}, x_{j+2}) \\ p(x_{j+4})\left(p(x_{j+1})q(x_{j+3}) + q(x_{j+4})p(x_{j+2})\right) + p(x_j)q^2(x_{j+3}), & x \in [x_{j+2}, x_{j+3}) \\ p^3(x_{j+4}), & x \in [x_{j+3}, x_{j+4}] \end{cases}$$

where,

$$q(x_j) = \sin\left(\frac{x - x_j}{2}\right),\ p(x_j) = \sin\left(\frac{x_j - x}{2}\right),\ z = \sin\left(\frac{h}{2}\right)\sin(h)\sin\left(\frac{3h}{2}\right)$$

where $h = (b-a)/n$ and $T_j^4(x)$ is a piecewise cubic trigonometric function with some geometric properties like C^2 continuity, non-negativity and partition of unity [16,17]. The values of $T_j^4(x)$ and its derivatives at nodal points are required and these derivatives are tabulated in Table 1. Secondly, we discuss the cubic trigonometric B-spline collocation method (CuTBSM) for the solving numerically the coupled viscous Burgers' system (1)

Table 1: Values $T_j^4(x)$ and its derivatives.

x	x_j	x_{j+1}	x_{j+2}	x_{j+3}	x_{j+4}
T_j	0	b_1	b_2	b_1	0
T_j'	0	b_3	0	b_4	0
T_j''	0	b_5	b_6	b_5	0

where

$$b_1 = \frac{\sin^2\left(\frac{h}{2}\right)}{\sin(h)\sin\left(\frac{3h}{2}\right)},$$

$$b_2 = \frac{2}{1+2\cos(h)},$$

$$b_3 = -\frac{3}{4\sin\left(\frac{3h}{2}\right)},$$

$$b_4 = \frac{3}{4\sin\left(\frac{3h}{2}\right)},$$

$$b_5 = \frac{3\left(1+3\cos(h)\right)}{16\sin^2\left(\frac{h}{2}\right)\left(2\cos\left(\frac{h}{2}\right)+\cos\left(\frac{3h}{2}\right)\right)},$$

$$b_6 = -\frac{3\cos^2\left(\frac{h}{2}\right)}{\sin^2\left(\frac{h}{2}\right)\left(2+4\cos(h)\right)}.$$

Description of Numerical Method

This section discusses the cubic trigonometric B-spline collocation method for solving numerically the couple viscous Burgers' equation (1). The solution domain $a \le x \le b$ is equally divided by knots x_j into N subintervals $[x_j, x_{j+1}]$, $j = 0,1,2,...,N-1$ where $a = x_0 < x_1 < ... < x_N = b$.

Our approach for couple burgers' system (1) using cubic trigonometric B-spline is to seek an approximate solution as

$$U_j(x,t) = \sum_{j=-3}^{N-1} C_j(t) T_j^4(x)$$
$$V_j(x,t) = \sum_{j=-3}^{N-1} D_j(t) T_j^4(x) \qquad (5)$$

where $C_j(t)$ and $D_j(\mathrm{t})$ are to be determined for the approximated solutions $V_j(x,t), U_j(x,t)$ to the exact solutions $u_{exc}(x,t), v_{exc}(x,t)$ at the point (x_j, t_i) .

The approximations U_j^i, V_j^i at the point (x_j, t_i) over subinterval $[x_j, x_{j+1}]$ can be defined as

$$U_j^i = \sum_{k=j-3}^{j-1} C_k^i T_k^4(x)$$
$$V_j^i = \sum_{k=i-3}^{i-1} D_k^j T_k^4(x) \qquad (6)$$

where $j = 0,1,2,...,N$. So as to get the approximations to the solution, the values of $T_j^4(x)$ and its derivatives at nodal points are required and these derivatives are tabulated using approximate functions (4) and (6), the values at the knots of U_j^i and V_j^i and their derivatives up to second orders are

$$\begin{cases} (U)_j^i = b_1 C_{j-3}^i + b_2 C_{j-2}^i + b_1 C_{j-1}^i, \\ \left(\frac{\partial U}{\partial x}\right)_j^i = b_3 C_{j-3}^i + b_4 C_{j-1}^i \\ \left(\frac{\partial^2 U}{\partial x^2}\right)_j^i = b_5 C_{j-3}^i + b_6 C_{j-2}^i + b_5 C_{j-1}^i \\ (V)_j^i = b_1 D_{j-3}^i + b_2 D_{j-2}^i + b_1 D_{j-1}^i, \\ \left(\frac{\partial V}{\partial x}\right)_j^i = b_3 D_{j-3}^i + b_4 D_{j-1}^i, \\ \left(\frac{\partial^2 V}{\partial x^2}\right)_j^i = b_5 C_{j-3}^i + b_6 C_{j-2}^i + b_5 C_{j-1}^i, \end{cases} \qquad (7)$$

The approximations for the solutions of couple viscous Burgers' equation (1) at t_{j+1} th time level can be given as

$$(U_t)_j^i + \theta A_j^{i+1} + (1-\theta) A_j^{i+1} = 0$$
$$(V_t)_j^i + \theta B_j^{i+1} + (1-\theta) B_j^{i+1} = 0 \quad (8)$$

where $A_j^i = \alpha (UU_x)_j^i + \beta ((UV)_x)_j^i - (U_{xx})_j^i$ and $B_j^i = \alpha (VV_x)_j^i + \eta ((UV)_x)_j^i - (V_{xx})_j^i$ the subscripts j and $j+1$ are successive time levels, $j = 0,1,2,3,\ldots$ Discretizing the time derivatives in the usual finite difference way and rearranging the equations, we get

$$U_j^{i+1} + \Delta t \theta A_j^{i+1} = U_j^i - \Delta t (1-\theta) A_j^i$$
$$V_j^{i+1} + \Delta \theta t B_j^{i+1} = V_j^j - \Delta t (1-\theta) \mathrm{B}_j^i \quad (9)$$

where Δt is the time step size. The nonlinear term $(UU_x)_i^{j+1}$, $(VV_x)_i^{j+1}$ and $((UV)_x)_i^{j+1}$ in equation (9) is linearized by using the following form[18]:

$$((UV)_x)^{i+1} = (VU_x)^{i+1} + (UV_x)^{i+1}$$
$$(UU_x)^{j+1} = U^{i+1} U_x^i + U^i U_x^{i+1} - U^{i+1} U_x^i \quad (10)$$
$$(VV_x)^{i+1} = V^{i+1} V_x^i + V^i V_x^{i+1} - V^{i+1} V_x^i$$

Substituting equation (10) into (9) and for Crank-Nicolson scheme [19]we set $\theta = 0.5$ in this paper. The equation (10) yields it the following

$$(1 + \frac{\Delta t}{2}\alpha U_x^i + \frac{\Delta t}{2}\beta V_x^i) U^{i+1} + (\frac{\Delta t}{2}\alpha U^i + \frac{\Delta t}{2}\beta V^i) U_x^{i+1} + \frac{\Delta t}{2}\beta U^i V_x^{i+1} + \frac{\Delta t}{2}\beta V^{i+1} U_x^i - \frac{\Delta t}{2} U_{xx}^{i+1} = U^i + U_{xx}^i$$
$$(1 + \frac{\Delta t}{2}\alpha V_x^i + \frac{\Delta t}{2}\beta U_x^i) V^{i+1} + (\frac{\Delta t}{2}\alpha V^i + \frac{\Delta t}{2}\eta U^i) V_x^{i+1} + \frac{\Delta t}{2}\eta V^i U_x^{i+1} + \frac{\Delta t}{2}\eta U^{i+1} V_x^i - \frac{\Delta t}{2} V_{xx}^{i+1} = V^i + V_{xx}^i$$

(11)

After simplifying (11) and using (7), the system consists $2(N+1)$ linear equations known with $2(N+3)$ unknowns $C_{-3}, C_{-2}, \ldots, C_{N-1}, D_{-3}, D_{-2}, \ldots, D_{N-1}$ at the time level $t = t_{j+1}$.

The boundary conditions given in (3) are applied for four additional linear equations to get a unique solution of the resulting system.

$$(U)_0^{i+1} = f_1(t_{j+1})$$
$$(U)_N^{i+1} = f_2(t_{j+1})$$
$$(V)_0^{i+1} = g_1(t_{j+1}) \quad (12)$$
$$(V)_N^{i+1} = g_2(t_{j+1})$$

Thus, the system becomes a matrix system of dimension $2(N+3) \times 2(N+3)$ which is a tri-diagonal system that can be solved by the Thomas Algorithm [16-17]. The system (11) can be written in the matrix form as follows:

$$MF^{i+1} = NF^i + b \qquad (13)$$

where

$$F^i = [C_{-3}^i, C_{-2}^i, \ldots, C_{N-1}^i, D_{-3}^i, D_{-2}^i, \ldots, D_{N-1}^i]^T$$
$$b = [f_1(t_{j+1}), 0, 0, 0, .. f_2(t_{j+1}), g_1(t_{j+1}), 0, 0, 0, .. g_2(t_{j+1})]^T, \quad j =$$

and M is an $2(N+3) \times 2(N+3)$ dimensional matrix given by

$$
M=\left[\begin{array}{ccccccc|ccccccc}
b_1 & b_2 & b_1 & 0 & . & . & 0 & 0 & 0 & 0 & . & . & . & 0 \\
a & b & c & 0 & . & . & 0 & d & e & f & . & . & . & 0 \\
0 & a & b & c & . & . & 0 & 0 & 0 & 0 & . & . & . & 0 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & . & . & a & b & c & 0 & 0 & 0 & . & d & e & f \\
0 & . & . & . & b_1 & b_2 & b_1 & 0 & 0 & 0 & . & . & . & 0 \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - \\
0 & 0 & . & . & . & . & 0 & b_1 & b_2 & b_1 & 0 & . & . & 0 \\
d_1 & e_1 & f_1 & . & . & . & 0 & a_1 & k & c_1 & 0 & . & . & 0 \\
. & . & . & . & . & . & 0 & 0 & 0 & . & . & . & . & . \\
. & . & . & . & . & , & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & . & . & d_1 & e_1 & f_1 & 0 & 0 & . & . & a_1 & k & c_1 \\
0 & 0 & . & . & . & 0 & 0 & 0 & 0 & . & . & b_1 & b_2 & b_1
\end{array}\right].
$$

Also N is a $2(N+3)\times 2(N+3)$ dimensional matrix given by

$$
N=\left[\begin{array}{ccccccc|ccccccc}
0 & 0 & 0 & 0 & . & . & 0 & 0 & 0 & 0 & . & . & . & 0 \\
p_1 & q_1 & z_1 & 0 & . & . & 0 & 0 & 0 & 0 & . & . & . & 0 \\
0 & p_1 & q_1 & z_1 & . & . & 0 & 0 & 0 & 0 & . & . & . & 0 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & . & . & p_1 & q_1 & z_1 & 0 & 0 & 0 & . & 0 & 0 & 0 \\
0 & . & . & . & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - \\
0 & 0 & . & . & . & . & 0 & 0 & 0 & 0 & 0 & . & . & 0 \\
0 & 0 & 0 & . & . & . & 0 & p_2 & q_2 & z_2 & 0 & . & . & 0 \\
. & . & . & . & . & . & 0 & 0 & 0 & . & . & . & . & . \\
. & . & . & . & . & , & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & . & . & 0 & 0 & 0 & 0 & 0 & . & . & p_2 & q_2 & z_2 \\
0 & 0 & . & . & . & 0 & 0 & 0 & 0 & . & . & 0 & 0 & 0
\end{array}\right].
$$

where

$$a=(1+\frac{\Delta t}{2}\alpha u_x^i+\frac{\Delta t}{2}\beta v_x^i)b_1+(\frac{\Delta t}{2}\alpha u^i+\frac{\Delta t}{2}\beta v^i)b_3-\frac{\Delta t}{2}b_5,$$

$$b=(1+\frac{\Delta t}{2}\alpha u_x^i+\frac{\Delta t}{2}\beta v_x^i)b_2-\frac{\Delta t}{2}b_6,$$

$$c=(1+\frac{\Delta t}{2}\alpha u_x^i+\frac{\Delta t}{2}\beta v_x^i)b_1+(\frac{\Delta t}{2}\alpha u^i+\frac{\Delta t}{2}\beta v^i)b_4-\frac{\Delta t}{2}b_5,$$

$$d=\frac{\Delta t}{2}\beta u_x^i b_1+\frac{\Delta t}{2}\beta u^i b_3,$$

$$e=\frac{\Delta t}{2}\beta u_x^i b_2,$$

$$f=\frac{\Delta t}{2}\beta u_x^i b_1+\frac{\Delta t}{2}\beta u^i b_4,$$

$$p_1=b_1+\frac{\Delta t}{2}b_5,$$

$$q_1=b_2+\frac{\Delta t}{2}b_6,$$

$$z_1=b_1+\frac{\Delta t}{2}b_5,$$

$$a_1=(1+\frac{\Delta t}{2}\alpha v_x^i+\frac{\Delta t}{2}\eta u_x^i)b_1+(\frac{\Delta t}{2}\alpha v^i+\frac{\Delta t}{2}\eta u^i)b_3-\frac{\Delta t}{2}b_5,$$

$$k=(1+\frac{\Delta t}{2}\alpha v_x^i+\frac{\Delta t}{2}\eta u_x^i)b_2-\frac{\Delta t}{2}b_6,$$

$$c_1=(1+\frac{\Delta t}{2}\alpha v_x^i+\frac{\Delta t}{2}\eta u_x^i)b_1+(\frac{\Delta t}{2}\eta u^i+\frac{\Delta t}{2}\alpha v^i)b_4-\frac{\Delta t}{2}b_5,$$

$$d_1=\frac{\Delta t}{2}\eta v_x^i b_1+\frac{\Delta t}{2}\eta u^i b_3,$$

$$e_1=\frac{\Delta t}{2}\eta v_x^i b_2,$$

$$f_1=\frac{\Delta t}{2}\eta v_x^i b_1+\frac{\Delta t}{2}\eta u^i b_4,$$

$$p_2=b_1+\frac{\Delta t}{2}b_5,$$

$$q_2=b_2+\frac{\Delta t}{2}b_6,$$

$$z_2=b_1+\frac{\Delta t}{2}b_5.$$

Initial State

The initial vectors D^0 and C^0 are computed from the initial conditions, the approximate solution U_j^{i+1} *and* V_j^{i+1} at a particular time can be calculated repeatedly the recurrence relation. D^0, C^0 can be provided from initial condition and boundary values of the derivatives as follows:

$$\begin{cases}(U_j^0)_x=u_0'(x_j) & j=0\\ U_j^0=u_0(x_j) & j=0,1,\dots,N\\ (U_j^0)_x=u_0'(x_j) & j=N\end{cases} \quad (14)$$

also to approximate another solution V_j^{i+1}

$$\begin{aligned}(V_j^0)_x&=v_0'(x_j) & j&=0\\ V_j^0&=v_0(x_j) & j&=0,1,\dots,N \quad (15)\\ (V_j^0)_x&=v_0'(x_j) & j&=N\end{aligned}$$

Thus, equations (14) and (15) provided a $2(N+3)\times 2(N+3)$ matrix system, of the form

$$AF^0=d$$

where

$$A=\left[\begin{array}{ccccccc|ccccccc}
b_3 & 0 & b_4 & 0 & . & . & 0 & 0 & 0 & 0 & . & . & . & 0\\
b_1 & b_2 & b_1 & 0 & . & . & 0 & 0 & 0 & 0 & . & . & . & 0\\
0 & b_1 & b_2 & b_1 & . & . & 0 & 0 & 0 & 0 & . & . & . & 0\\
. & . & . & . & . & . & . & . & . & . & . & . & . & .\\
. & . & . & . & . & . & . & . & . & . & . & . & . & .\\
. & . & . & . & . & . & . & . & . & . & . & . & . & .\\
0 & 0 & . & . & b_1 & b_2 & b_1 & 0 & 0 & 0 & . & . & . & 0\\
0 & . & . & . & b_3 & 0 & b_4 & 0 & 0 & 0 & . & . & . & 0\\
- & - & - & - & - & - & - & - & - & - & - & - & - & -\\
0 & 0 & . & . & . & . & 0 & b_3 & 0 & b_4 & 0 & . & . & 0\\
0 & 0 & . & . & . & . & 0 & b_1 & b_2 & b_1 & 0 & . & . & 0\\
. & . & . & . & . & . & 0 & 0 & 0 & . & . & . & . & .\\
. & . & . & . & . & , & . & . & . & . & . & . & . & .\\
. & . & . & . & . & . & . & . & . & . & . & . & . & .\\
. & . & . & . & . & . & . & . & . & . & . & . & . & .\\
0 & 0 & . & . & . & 0 & 0 & 0 & 0 & . & . & b_1 & b_2 & b_1\\
0 & 0 & . & . & . & 0 & 0 & 0 & 0 & . & . & b_3 & 0 & b_4
\end{array}\right].$$

$$F^0=[C_{-3}^0,C_{-2}^0,...,C_{N-1}^0,D_{-3}^0,D_{-2}^0,...,D_{N-1}^0]^T$$

$$d=[u_0'(x_0),u_0(x_0),u_0(x_1),...u_0(x_N),u_0'(x_N),v_0'(x_0),v_0(x_0),v_0(x_1),...v_0(x_N),v_0'(x_N)]^T$$

Stability Analysis

We have investigated stability of the proposed method by applying von-Neumann method. To apply this method, we have linearized the nonlinear terms $(UV)_x$ in the equation (1) by considering U and V as a local constants μ_1 and μ_2 respectively. Substituting the approximate solution for u and v and their derivatives at the knots in the modified equation after discretizing the time derivative in the usual finite difference way and applying Crank–Nicolson scheme yields a difference equation with the variables given as:

$$a_1C_{j-3}^{i+1}+a_2C_{j-2}^{i+1}+a_3C_{j-1}^{i+1}+a_4D_{j-3}^{i+1}+a_5D_{j-1}^{i+1}=a_6C_{j-3}^{i+1}+a_7C_{j-2}^{i+1}+a_8C_{j-1}^{i+1}+a_9D_{j-3}^{i+1}+a_{10}D_{j-1}^{i+1} \quad (16)$$

Now on substituting $C^n=A\delta^n\exp(\mathrm{imp}\varphi\mathrm{h})\,\mathrm{and}\,\mathrm{D}^n=B\delta^n\exp(\mathrm{imp}\varphi\mathrm{h})$ in the equation where A and B are harmonies amplitude φ is the mode number, h is the element sizes and $i^2=-1$

$$\begin{cases}a_1A\delta^{n+1}\exp(i(m-3)p\varphi h)+a_2A\delta^{n+1}\exp(i(m-2)p\varphi h)+a_3A\delta^{n+1}\exp(i(m-1)p\varphi h)\\ +a_4B\delta^{n+1}\exp(i(m-3)p\varphi h)+a_5B\delta^{n+1}\exp(i(m-1)p\varphi h)\end{cases}$$

$$=\begin{cases}a_6A\delta^{n}\exp(i(m-3)p\varphi h)+a_7A\delta^{n}\exp(i(m-2)p\varphi h)+a_8A\delta^{n}\exp(i(m-1)p\varphi h)+a_9B\delta^{n}\exp(i(m-3)p\varphi h)\\ a_{10}B\delta^{n}\exp(i(m-1)p\varphi h)\end{cases}$$

Where

$$a_1 = b_1 + \frac{\Delta t}{2}(\alpha\mu_1 + \beta\mu_2)b_3 - \frac{\Delta t}{2}b_5,$$

$$a_2 = b_2 - \frac{\Delta t}{2}b_6,$$

$$a_3 = b_1 + \frac{\Delta t}{2}(\alpha\mu_1 + \beta\mu_2)b_4 - \frac{\Delta t}{2}b_5,$$

$$a_4 = \frac{\Delta t}{2}\beta\mu_1 b_3,$$

$$a_5 = \frac{\Delta t}{2}\beta\mu_1 b_4,$$

$$a_6 = b_1 - \frac{\Delta t}{2}(\alpha\mu_1 + \beta\mu_2)b_3 + \frac{\Delta t}{2}b_5,$$

$$a_7 = b_2 + \frac{\Delta t}{2}b_6,$$

$$a_8 = b_1 - \frac{\Delta t}{2}(\alpha\mu_1 + \beta\mu_2)b_4 + \frac{\Delta t}{2}b_5,$$

$$a_9 = -\frac{\Delta t}{2}\beta\mu_1 b_3,$$

$$a_{10} = -\frac{\Delta t}{2}\beta\mu_1 b_4.$$

Now divide the both side on $\exp(i(m-2)p\varphi h)$ we obtain

$$\left|X_1 + iY\right|\delta^{i+1} = \left|X_2 + iY\right|\delta^i \qquad (17)$$

where

$$X_1 = A((2b_1\cos(\varphi\mathrm{h}) + \mathrm{b}_2) - \Delta\,\mathrm{t}(\frac{1}{2}b_6 + b_5\cos(\varphi\mathrm{h}))\,,$$

$$X_2 = A(2b_1\cos(\varphi\mathrm{h}) + b_2) + \Delta t(b_5\cos(\varphi\mathrm{h}) + \frac{1}{2}b_6)\,,$$

$$Y = \{A\Delta t\,(\alpha\mu_1 + \beta\mu_2) + \Delta\,\mathrm{t}\,\beta\mu_1)b_4\sin(\varphi\mathrm{h})$$

$$|\delta| = \sqrt{\left(\frac{X_1X_2 + Y^2}{X_1^2 + Y^2}\right)^2 + \left(\frac{X_1Y + X_2Y}{X_1^2 + Y^2}\right)^2} \le 1$$

(18)

From (18) the coupled Burgers' equation is unconditionally stable since the modulus of the Eigen-values is less thanor equal to one. This means that there is on restriction on grid size, i.e. $h\,and\,\Delta t,$ and step size in time level but we should choose those values of $h\,and\,\Delta t,$ for which we get the best accuracy of the scheme.

Numerical Illustrations

To test the accuracy of proposed method, two examples are given in this section with L_∞ and relative L_2 error norms are calculated by

$$L_\infty = \max_i\left|U_{exci} - U_i\right|,\quad L_2 = \frac{\sqrt{\sum_i^N\left|U_{exci} - U_i\right|^2}}{\sqrt{\sum_i^N\left|U_{exci}\right|^2}}$$

The numerical order of convergence p for numerical solution $U(x,t)$ and $V(x,t)$ is obtained by using the formula [9, 17]

$$p = \frac{Log\left(L_\infty(N)/L_\infty(2N)\right)}{Log\left(2N/N\right)}$$

where $L_\infty(N)$ and $L_\infty(2N)$ are the errors at number of partitions n and $2n$ respectively. We compare the numerical solutions obtained by cubic trigonometric B-spline collocation method for coupled viscous Burgers' equation (1) with known exact solutions and those numerical methods which were exiting in literature. Numerical results are computed by cubic trigonometric B-spline collocation method for coupled viscous Burgers' equation (1) at different time levels which are tabulated and depicted in different Tables and Figures respectively. The feasibility of the method is shown by test problems and the approximated solutions are found to be in good agreement with the exact solutions. The proposed method is superior to Mittal and Arora [9], Rashid et al. [12], Khater et al. [20] and Rashid et al. [21].

Problem 1:

Consider the one dimensional coupled viscous Burgers' equation (1) with $\alpha = \beta = 1.0$ and $\eta = -2$ which leads equation (1) – (2) as [9, 12, 21]:

$$\begin{cases} u_t - u_{xx} - 2uu_x + (uv)_x = 0 \\ v_t - v_{xx} - 2vv_x + (uv)_x = 0 \end{cases}$$

with the initial conditions given by

$$u_0(x) = v_0(x) = \sin(x), \ -\pi \le x \le \pi$$

and boundary conditions as follows:

$$f_1(t) = f_2(t) = 0, \ g_1(t) = g_2(t) = 0, \quad 0 < t \le T$$

The known solutions of this problem are $U_{exc}(x,t) = V_{exc}(x,t) = e^{-t}\sin(x)$. The proposed method is applied to calculate the numerical solutions of coupled viscous Burgers' equation (1)-(3) by taking domain $-\pi \le x \le \pi$ with $\Delta t = 0.001$. The absolute errors at different time levels and different number of partitions are reported in Table 2. The ratio in absolute errors L_∞ and order of convergence of the proposed method at different time levels and different number of partitions which are tabulated in Table 3 and it shows that the method has an approximately two order of convergence. Figure 1 depicts the graphs of comparison between exact and numerical solutions at different time levels with $N = 200, \Delta t = 0.001$. Figure 2 shows the space-time graph of exact and approximate solutions at $T = 1.0$ with $h = 1/200, \Delta t = 0.001$. Due to symmetric initial and boundary conditions, the numerical results are similar for $V(x,t)$. The numerical proposal method gives more accurate than Mittal and Arora [9] and Rashid et al. [12, 21].

Table 2: Relative errors and maximum errors of Problem 1 for $U(x,t)$ with $\Delta t = 0.001$

time	CuTBSM (Proposed)				CuBSM[9]			
	L_2 (N=200)	L_∞	L_2 (N=400)	L_∞	L_2 (N=200)	L_∞	L_2 (N=400)	L_∞
0.1	1.23E-05	6.96E-06	1.91E-06	1.73E-06	8.21E-06	7.45E-06	2.05E-06	1.86E-06
0.5	3.85E-05	2.33E-05	9.59E-06	5.82E-06	2.49E-05	4.10E-05	1.02E-05	6.22E-06
1.0	7.70E-05	2.83E-05	1.91E-05	7.06E-06	3.00E-05	8.21E-05	2.04E-05	7.56E-06

time	Rashid [12]				Rashid [21]			
	L_2 (N=200)	L_∞	L_2 (N=400)	L_∞	L_2 (N=200)	L_∞	L_2 (N=200)	L_∞
0.1	No data	No data	No data	No data	No data	No data	No data	No data
0.5								
1.0	2.88E-05	1.16E-05			2.77E-05	1.05E-05		

Table 3: L_∞ errors, ratio and order of convergence of Problem 1 for $U(x,t)$ at different time levels

Method	N	$L_\infty(t=0.1)$	Ratio	Order of Conv.	$L_\infty(0.5)$	Ratio	Order of Conv.
CuTBSM	32	2.7336E-04	--------	-------	9.1674E-04	--------	-------
	64	6.8117E-05	4.0090	2.0034	2.2854E-04	4.0113	2.0440
	128	1.7029E-05	4.0036	2.0013	5.7076E-05	4.0040	2.0015
	256	4.2510E-06	4.0058	2.0021	1.4247E-05	4.0061	2.0022
	512	1.0570E-06	4.0215	2.0077	3.5428E-06	4.0216	2.0077
CuBSM[9]	32	2.9104E-04	--------	-------	9.7478E-04	--------	-------
	64	7.2704E-05	4.0030	2.001	2.4361E-04	4.0014	2.005
	128	1.8178E-05	3.9996	1.999	6.0896E-05	4.0004	2.001
	256	4.5497E-05	3.9953	1.998	1.5223E-05	4.0003	2.001
	512	1.1430E-06	3.9806	1.993	3.8052E-05	4.0006	2.002

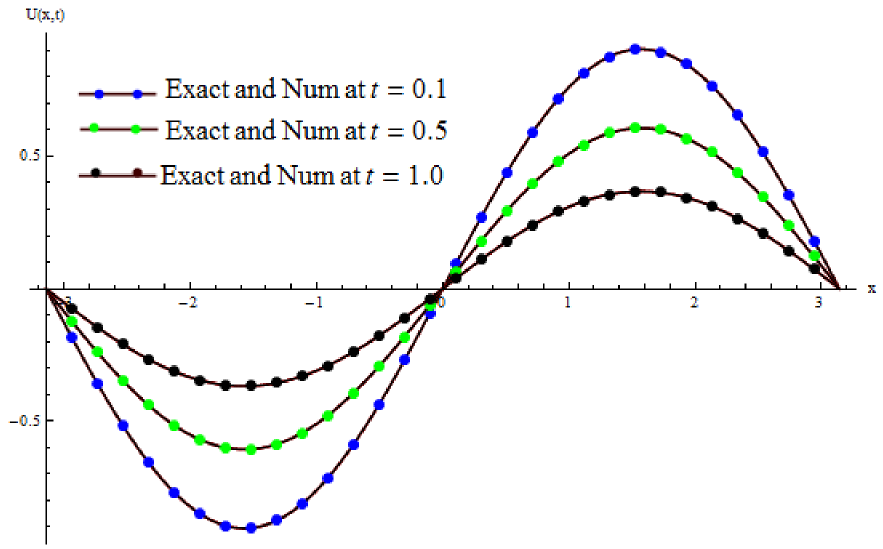

Figure 1: A comparison between numerical and exact solutions of $U(x,t)$ for Problem 1

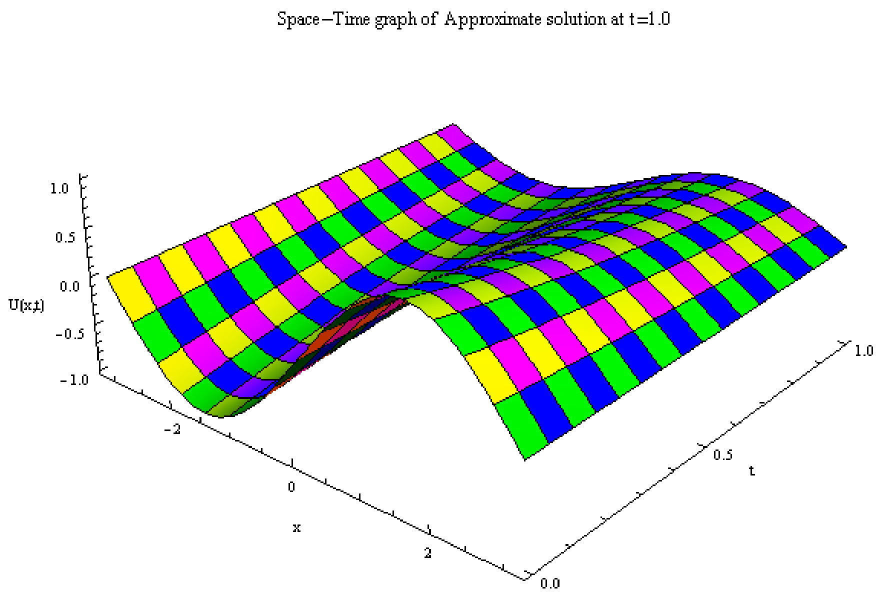

Figure 2: Space-time graph of approximate solution $U(x,t)$ for Problem 1 at $t=1.0$ and $\Delta t=0.001$

Problem 2:

Consider the one dimensional coupled viscous Burgers' equation (1) for different values of α,β and $\eta=2$ which leads equation (1) – (2) as [9, 12, 20-21]:

$$\begin{cases} u_t - u_{xx} + 2uu_x + \alpha(uv)_x = 0 \\ v_t - v_{xx} + 2vv_x + \beta(uv)_x = 0 \end{cases}$$

With the initial conditions given by

$$\begin{cases} u_0(x) = a_0\left(1-\tanh(\lambda x)\right) \\ v_0(x) = a_0\left(\left(\dfrac{2\beta-1}{2\alpha-1}\right)-\tanh(\lambda x)\right), \end{cases} \quad -10 \le x \le 10$$

and boundary conditions as follows:

$$\begin{cases} f_1(t) = a_0\left(1-\tanh\left(\lambda(-10-2\lambda t)\right)\right) \\ f_2(t) = a_0\left(1-\tanh\left(\lambda(10-2\lambda t)\right)\right) \end{cases} \quad 0 < t \le T$$

and

$$\begin{cases} g_1(t) = a_0\left(\left(\frac{2\beta-1}{2\alpha-1}\right)-\tanh\left(\lambda(-10-2\lambda t)\right)\right) \\ g_2(t) = a_0\left(\left(\frac{2\beta-1}{2\alpha-1}\right)-\tanh\left(\lambda(10-2\lambda t)\right)\right) \end{cases} \quad 0 < t \le T$$

where $a_0 = 0.05$ and $\lambda = \frac{a_0}{2}\left(\frac{4\alpha\beta-1}{2\alpha-1}\right)$ .The known solutions of this problem are $U_{exc}(x,t) = a_0\left(1-\tanh\left(\lambda(x-2\lambda t)\right)\right)$, $V_{exc}(x,t) = a_0\left(\left(\frac{2\beta-1}{2\alpha-1}\right)-\tanh\left(\lambda(x-2\lambda t)\right)\right)$. The proposed method is used to calculate the numerical solutions of coupled viscous Burgers' equation (1)-(3) over the domain $-10 \le x \le 10$ with $\Delta t = 0.01$, $N = 100$. The absolute errors at different time levels and different values of $\alpha,\ \beta$ for $U(x,t)$ and $V(x,t)$ are tabulated in Table 4 and Table 5 respectively. Figures 3 and 4 show the space-time graph of exact and approximate solutions $U(x,t)$ and $V(x,t)$ at $T = 1.0$ with $h = 0.01, \Delta t = 0.01$. The numerical results of this problem are more accurate than Mittal and Arora [9], Rashid et al. [12, 21] and Khater [20].

Table 4: Relative errors and maximum errors of Problem 2 for $U(x,t)$ with $\Delta t = 0.01$

t	α	β	CuTBSM L_2	L_∞	CuBSM[9] L_2	L_∞	Khater[20] L_2	L_∞	Rashid[12] L_2	L_∞
0.5	0.10	0.30	3.21E-05	1.21E-05	6.74E-04	4.17E-05	1.44E-03	4.38E-05	3.25E-05	9.62E-04
	0.30	0.03	1.98E-05	6.67E-05	7.33E-04	4.59E-05	6.68E-03	4.58E-05	2.73E-05	4.31E-04
1.0	0.10	0.30	6.26E-05	2.33E-05	1.33E-03	8.26E-05	1.27E-03	8.66E-05	2.40E-05	1.15E-03
	0.30	0.03	3.89E-04	1.32E-05	1.45E-03	9.18E-05	1.30E-03	9.16E-05	2.83E-05	1.27E-03
Rashid[21]										
0.5	0.10	0.10							1.27E-05	3.26E-05
	0.30	0.30							1.16E-05	3.23E-05
1.0	0.10	0.10							1.14E-05	2.27E-05
	0.30	0.30							1.13E-05	2.17E-05

Table 5: Relative errors and maximum errors of Problem 2 for $V(x,t)$ with $\Delta t = 0.01$

t	α	β	CuTBSM L_2	L_∞	CuBSM[9] L_2	L_∞	Khater[20] L_2	L_∞	Rashid[12] L_2	L_∞
0.5	0.10	0.30	5.75E-05	1.66E-05	9.05E-04	1.48E-04	5.42E-04	4.99E-05	2.74E-05	3.33E-04
	0.30	0.03	2.44E-05	9.65E-05	1.59E-03	5.73E-04	1.20E-03	1.81E-04	2.45E-04	1.15E-03
1.0	0.10	0.30	1.13E-05	3.28E-05	1.25E-03	4.77E-05	1.29E-03	9.92E-05	3.74E-05	1.16E-03
	0.30	0.03	4.73E-	1.86E-05	2.25E-03	3.62E-04	2.35E-03	3.62E-04	4.52E-04	1.64E-03

			04							
Rashid[21]										
0.5	0.10	0.30							1.12E-05	1.27E-05
	0.30	0.03							1.17E-05	1.26E-05
1.0	0.10	0.30							1.12E-05	1.21E-05
	0.30	0.03							1.13E-05	1.22E-05

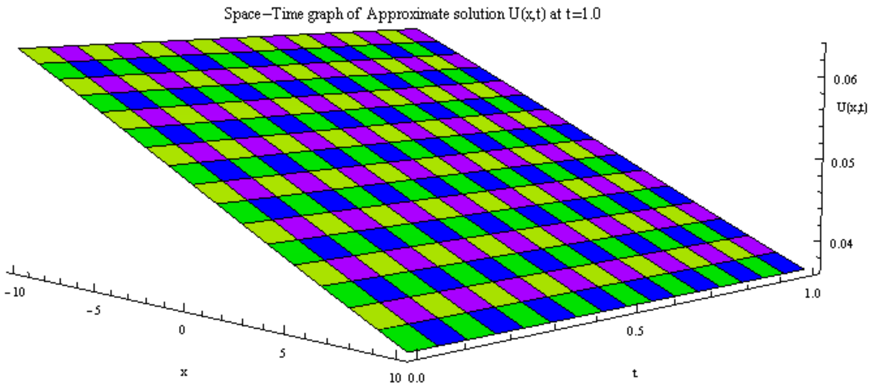

Figure 3: Space-time graph of approximate solution $U(x,t)$ for Problem 2 at $t = 1.0$ and $\Delta t = 0.01$

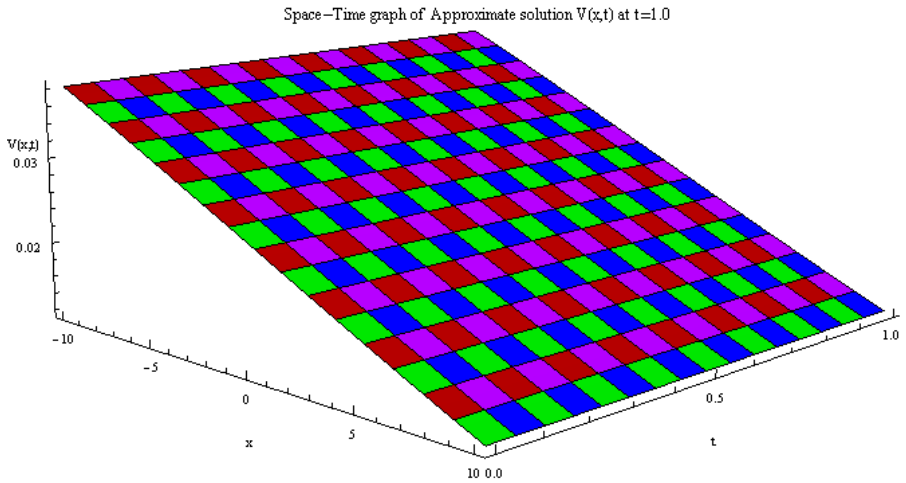

Figure 4: Space-time graph of approximate solution $V(x,t)$ for Problem 2 at $t = 1.0$ and $\Delta t = 0.01$

Conclusions

This paper has investigated the application of cubic trigonometric B-spline collocation method to find the numerical solution of the one dimensional coupled viscous Burgers' equation with initial condition and Dirichlet boundary conditions. A usual finite difference approach is used to discretize the time derivatives. The cubic trigonometric B-spline is used for interpolating the solutions at each time. The numerical results shown in Tables 2-5 and Figures 1-4 indicate the reliability of results obtained. The obtained solution to the coupled viscous Burgers' equation for various time levels has been compared with the exact solution and existing methods by calculating L_∞ and L_2. It is found that cubic trigonometric B-spline collocation approach has provided more accurate results as compared to Mittal and Arora [9], Rashid et al. [12, 21] and Khater [20].

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