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Unsteady Boundary Layer Flow and Symmetry Analysis of a Carreau Fluid

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Abstract

Two dimensional unsteady boundary layer equations of a non-Newtonian Carreau fluid are considered in the present paper. Symmetries of boundary layer equations are found by using Lie group theory. Using scaling and translation symmetries, unsteady boundary layer equations and boundary conditions are transformed into a partial differential system with two variables. Lie Groups are further applied to these equations. Using infinitesimal generators of these equations, the boundary value problem is transformed into an ordinary differential equations system, which is numerically integrated under the boundary condition. The boundary condition is a moving surface, with suction injection. Effects of power-law index and Carreau fluid coefficient on the boundary layers are shown.

Keywords: Carreau fluid equations; Unsteady boundary layer flow; Lie Groups.

1. Introduction

The flow of an incompressible non-Newtonian fluid over a moving surface with suction or injection has important industrial applications, for example in the extrusion of a polymer sheet from a die or in the drawing of plastic films. During the manufacture of these sheets, the melt issues from a slit and is subsequently stretched to achieve the desired thickness. Materials manufactured by extrusion processes and heat-treated materials traveling between a feed roll and wind-up roll or on conveyor belts possess the characteristics of a moving continuous surface. The mechanical properties of the final product strictly depend on the stretching rate and on the rate of cooling in this process.

Boundary layer theory has been successfully applied to non-Newtonian fluids of various models. Among the many proposed models, an interesting one is due to Hansen Na [1]. They first considered a model in which the shear stress is an arbitrary function of the velocity gradient. This model is a classification of various types of visco-inelastic non-Newtonian fluids including Newtonian, Power-law, Williamson, Prandtl, Carreau and Powel-Eyring fluids as special cases. The work on boundary layers of power-law fluids was due to Yurusoy[2]. Yurusoy [2] examined the unsteady boundary layer flow of power-law fluids by using a special similarity transformation. Very recently, a symmetry analysis was presented for boundary layer equations of the modified second grade fluid [3, 4]. In contrast to power-law model the Carreau viscosity model is one of the non-Newtonian fluid models in which constitutive relationship is valid for low and high shear rates. Due to this reason it has gained wider acceptance in industrial and technological flows.

Hsu et al. [5] investigated electrophoresis of a sphere at an arbitrary position in a spherical cavity filled with Carreau fluid. Ali and Hayat [6] discussed the peristaltic transport of Carreau fluid in an asymmetric channel. Shamekhi and Sadeghy [7] analyzed the cavity flow simulation of Carreau-Yasuda fluid using PIM meshfree method. Tshehla [8] studied the flow of Carreau fluid down an inclined free surface. Olajuwon [9] studied convective heat and mass transfer in a hydromagnetic flow Carreau fluid past a vertical porous plat in presence of thermal radiation and thermal diffusion. Three-dimensional peristaltic motion of Carreau fluid in a rectangular duct was described by Ellahi et al. [10].Pantokraturas [11] considered the Blasius and Sakiadis flow of a non-Newtonian Carreau fluid.

In this paper, two dimensional, unsteady boundary layer flows over a moving surface,with suction or injection for Carreau fluid is considered. Two-dimensional unsteady boundary layer flow of Carreau fluid has never been considered before. Continuity, momentum and energy equations are written and cast into a non-dimensional form. Boundary conditions are selected a moving surface, with suction or injection form. Lie Group theory is applied to the equations. The partial differential system with three independent variables is converted into an ordinary differential system via application of two successive symmetry generators. The ordinary differential equations are solved numerically. Effects of flow parameters on the development of momentum boundary layers are discussed.

2. Equations of Motion

The constitutive equation for a Carreau fluid is given as below [12]

$$\tau = \mu_\infty + (\mu_0 - \mu_\infty)\left[1 + (\lambda\dot{\gamma})^2\right]^{\frac{n-1}{2}}\dot{\gamma} \qquad (1)$$

where $\tau$is the stress tensor, μ_∞ is the infinity shear rate viscosity, μ_0 is the zero infinity shear rate viscosity, λ is time constant, n is the dimensionless power-law index, $\dot{\gamma}$ is strain rate tensor. The fluid is characterized as shear-thinning for 0<n<1, shear-thickening for n>1 and Newtonian for n=1. In the current formulation it is assumed that μ_∞ is zero. The continuity and momentum equations for Carreau non-Newtonian fluid model are.

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \qquad (2)$$

$$\begin{aligned}\rho\left(\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}\right) &= \mu_0\left[1 + \left(\lambda\frac{\partial \bar{u}}{\partial \bar{y}}\right)^2\right]^{\frac{n-1}{2}}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \\ &+ \mu_0(n-1)\left[1 + \left(\lambda\frac{\partial \bar{u}}{\partial \bar{y}}\right)^2\right]^{\frac{n-1}{2}}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)^2 + \rho\left(\frac{\partial \bar{U}}{\partial \bar{t}} + \bar{U}\frac{\partial \bar{U}}{\partial \bar{x}}\right)\end{aligned} \qquad (3)$$

Where $\bar{x}$ is the coordinate along the surface and $\bar{y}$ is the coordinate vertical to $\bar{x}$ $\bar{t}$ is time, $\bar{u}$ and $\bar{v}$ are the velocity components in $\bar{x}$ and $\bar{y}$, $\bar{U}(\bar{x},\bar{t})$ is the free stream velocity outside the boundary layer, $\rho$is the fluid density. The boundary conditions are selected to be in a general form. Special forms of these boundary conditions are

$$\bar{u}(\bar{x},0,\bar{t}) = \bar{A}(\bar{x},\bar{t}),\ \bar{v}(\bar{x},0,\bar{t}) = \pm\bar{V}(\bar{x},\bar{t}),\ \bar{u}(\bar{x},\infty,\bar{t}) = \bar{U}(\bar{x},\bar{t}) \qquad (4)$$

Where $\bar{A}(\bar{x},\bar{t})$ moving surface velocity, $\bar{V}(\bar{x},\bar{t})$ is the suction or injection velocity of the permeable surface. The non-dimensional form of equations (2)-(3) and boundary conditions are as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (5)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \left[1 + \left(\kappa\frac{\partial u}{\partial y}\right)^2\right]^{\frac{n-1}{2}}\frac{\partial^2 u}{\partial y^2} + \kappa De(n-1)\left[1 + \left(\kappa\frac{\partial u}{\partial y}\right)^2\right]^{\frac{n-1}{2}}\frac{\partial^2 u}{\partial y^2}\left(\frac{\partial u}{\partial y}\right)^2 + f(x,t) \qquad (6)$$

$$u(x,0,t) = A(x,t),\ v(x,0,t) = \pm V(x,t),\ u(x,\infty,t) = U(x,t) \qquad (7)$$

The non-dimensional parameters are related to the dimensional ones through the following relations:

$$x=\frac{\bar{x}}{L},\ y=\frac{\bar{y}}{L}\sqrt{Re},\ t=\frac{\bar{t}U_0}{L},\ u=\frac{\bar{u}}{U_0},\ v=\frac{\bar{v}}{U_0}\sqrt{Re},\ U=\frac{\bar{U}}{U_0},V=\frac{\bar{V}}{U_0}\sqrt{Re},\ A=\frac{\bar{A}}{U_0}$$
$$Re=\frac{\rho U_0 L}{\mu_0},\ f(x,t)=\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x},\kappa=\frac{\lambda U_0}{L},\ De=\frac{\rho\lambda U_0^2}{\mu_0} \tag{8}$$

Where Re is Reynolds number, De is Deborah number [11] and U_0 and L are the average free stream velocity, length of the horizontal surface, respectively.

3. Symmetry Reductions

The symmetry groups of equations (5) and (6) are found using classical Lie group approach [13, 14]. The one.-parameter infinitesimal Lie Group of transformations are defined as

$$\begin{aligned} x^* &= x+\varepsilon\xi_1(x,y,t,u,v)\\ y^* &= y+\varepsilon\xi_2(x,y,t,u,v)\\ t^* &= t+\varepsilon\xi_3(x,y,t,u,v)\\ u^* &= u+\varepsilon\eta_1(x,y,t,u,v)\\ v^* &= v+\varepsilon\eta_2(x,y,t,u,v)\end{aligned} \tag{9}$$

and the corresponding infinitesimal generator

$$X=\xi_1\frac{\partial}{\partial x}+\xi_2\frac{\partial}{\partial y}+\xi_3\frac{\partial}{\partial t}+\eta_1\frac{\partial}{\partial u}+\eta_2\frac{\partial}{\partial v} \tag{10}$$

Carrying out the straightforward algebra, we obtain the following results

$$\begin{aligned} &\xi_1=3ax+k_1(t),\ \ \xi_2=ay+k_2(x,t),\ \ \xi_3=2at+b\\ &\eta_1=au+k_1'(t),\ \ \eta_2=-av+u\frac{\partial k_2}{\partial x}+\frac{\partial k_2}{\partial t}\end{aligned} \tag{11}$$

and the form of f(x,t) is determined by the equation

$$h_1''(t)=af+\xi_1\frac{\partial f}{\partial x}+\xi_3\frac{\partial f}{\partial t} \tag{12}$$

The generators corresponding to infinitesimals in (11) can be written as follows

$$\begin{aligned} &X_1=\frac{\partial}{\partial t},\ X_2=3x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+2t\frac{\partial}{\partial t}+u\frac{\partial}{\partial u}-v\frac{\partial}{\partial v},\ \ X_3=k_1(t)\frac{\partial}{\partial x}+k_1'(t)\frac{\partial}{\partial u}\\ &X_4=k_2(x,t)\frac{\partial}{\partial y}+\left(u\frac{\partial k_2}{\partial x}+\frac{\partial k_2}{\partial t}\right)\frac{\partial}{\partial v}\end{aligned} \tag{13}$$

The above conditions restrict the infinitesimals (11) to the following form (see [13])

$$\begin{aligned}&\xi_1 = 3ax + k_1(t),\ \ \xi_2 = ay,\ \ \xi_3 = 2at + b\\ &\eta_1 = au + k_1'(t),\ \ \eta_2 = -av\end{aligned} \qquad (14)$$

The arbitrary functions A, V and U in boundary condition (7) satisfy the following equations

$$\begin{aligned}&aA + k_1{}'(t) = \xi_1 \frac{\partial A}{\partial x} + \xi_3 \frac{\partial A}{\partial t}\\ &-aV = \xi_1 \frac{\partial A}{\partial x} + \xi_3 \frac{\partial A}{\partial t}\\ &aU + k_1{}'(t) = \xi_1 \frac{\partial U}{\partial x} + \xi_3 \frac{\partial U}{\partial t}\end{aligned} \qquad (15)$$

If we take the scaling transformation a=1, $b=k_1=0$ and find the similarity variables and functions, we have

$$\begin{aligned}&\alpha = \frac{x}{t^{3/2}},\ \beta = \frac{y}{t^{1/2}},\ \ u = t^{1/2}F(\alpha,\beta),\ v = t^{-1/2}G(\alpha,\beta),\ U = t^{1/2}U(\alpha)\\ &V = t^{1/2}V(\alpha),\ A = t^{1/2}A(\alpha),\ \ f = t^{-1/2}f(\alpha),\end{aligned} \qquad (16)$$

Where f(α) and U(α) are related through the following equation

$$f = \frac{1}{2}U - \frac{3}{2}\alpha U' + UU' \qquad (17)$$

Then we substitute the new variables into the original equations (5)-(7), and obtain

$$F_\alpha + G_\beta = 0 \qquad (18)$$

$$\begin{aligned}\tfrac{1}{2}F - \tfrac{3}{2}\alpha F_\alpha - \tfrac{1}{2}\beta F_\beta + FF_\alpha + GF_\beta &= \left[1 + (\kappa F_\beta)^2\right]^{\frac{n-1}{2}} F_{\beta\beta}\\ &+ \kappa De(n-1)\left[1 + (\kappa F_\beta)^2\right]^{\frac{n-3}{2}} (F_\beta)^2 F_{\beta\beta} + f(\alpha)\end{aligned} \qquad (19)$$

$$F(\alpha,0) = A(\alpha),\ \ G(\alpha,0) = \pm V(\alpha),\ \ F(\alpha,\infty) = U(\alpha) \qquad (20)$$

On the other hand, if we consider translations in the x and t (a=0, $b=k_1=1$), we have

$$\begin{aligned}&\gamma = x - t,\ \lambda = y,\ u = P(\gamma,\lambda),\ v = Q(\gamma,\lambda),\ U = U(\gamma)\\ &V = V(\gamma),\ A = A(\gamma),\ f = f(\gamma),\ f(\gamma) = -U' + UU'\end{aligned} \qquad (21)$$

Substituting these variables into equations (5)-(7) we finally obtain the following partial differential system.

$$P_\gamma + Q_\lambda = 0 \qquad (22)$$

$$-P_\gamma + PP_\gamma + QP_\lambda = \left[1 + (\kappa P_\lambda)^2\right]^{\frac{n-1}{2}} P_{\lambda\lambda} + \kappa De(n-1)\left[1 + (\kappa P_\lambda)^2\right]^{\frac{n-3}{2}} (P_\lambda)^2 P_{\lambda\lambda} + f(\gamma) \qquad (23)$$

$$P(\gamma,0) = A(\gamma),\ \ Q(\gamma,0) = \pm V(\gamma),\ P(\gamma,\infty) = U(\gamma) \qquad (24)$$

3.1 Further Lie Group Analysis

In this section, we reduce the partial differential equations with to independent variables obtained to ordinary differential equations via Lie Groups. We first treat partial differential system (18)-(20). The infinitesimal generator

$$Y = \xi_1 \frac{\partial}{\partial \alpha} + \xi_2 \frac{\partial}{\partial \beta} + \eta_1 \frac{\partial}{\partial F} + \eta_2 \frac{\partial}{\partial G} \tag{25}$$

accepted by the equations has the form

$$\xi_1 = 2a,\ \xi_2 = 2h(\alpha),\ \eta_1 = 3a,\ \eta_2 = 2h'(\alpha)F + h(\alpha) - 3\alpha h'(\alpha),\ f = \frac{3}{4}\alpha + b \tag{26}$$

The boundary conditions in (20) restrict the infinitesimals (26) to the following form

$$\xi_1 = 2a,\ \xi_2 = 0,\ \eta_1 = 3a,\ \eta_2 = 2h'(\alpha)F + h(\alpha) - 3\alpha h'(\alpha),\ f = \frac{3}{4}\alpha + b \tag{27}$$

If we take a=1 and defined the similarity variables and functions, we have

$$\xi = \beta,\ F = \frac{3}{2}\alpha + S(\xi),\ G = R(\xi),\ A = \frac{3}{2}\alpha + a_1,\ V = a_2,\ U = \frac{3}{2}\alpha + a_3,\ a_3 = b/2 \tag{28}$$

Then we substitute the new variables into partial differentials equations system (18)-(20)

$$3 + 2R' = 0 \tag{29}$$

$$2S + S'\left(2R - \tfrac{1}{2}\xi\right) = \left[1 + (\kappa S')^2\right]^{\frac{n-1}{2}} S'' + \kappa De(n-1)\left[1 + (\kappa S')^2\right]^{\frac{n-3}{2}} (S')^2 S'' + 2a_3 \tag{30}$$

$$S(0) = a_1,\ R(0) = \pm a_2,\ S(\infty) = a_3 \tag{31}$$

Solving equations (29) together with boundary condition, we have

$$R = -\tfrac{3}{2}\xi \pm a_2 \tag{32}$$

Substituting equations (32) into equations (30), we finally obtain an ordinary differential equation with boundary condition

$$2S + S'\left(-2\xi \pm a_2\right) = \left[1 + (\kappa S')^2\right]^{\frac{n-1}{2}} S'' + \kappa De(n-1)\left[1 + (\kappa S')^2\right]^{\frac{n-3}{2}} (S')^2 S'' + 2a_3 \tag{33}$$

$$S(0) = a_1,\ \ S(\infty) = a_3 \tag{34}$$

Next, we apply Lie Groups to the partial differential equation system (22)-(24) obtained by translations. The generator has following form

$$Z = \xi_1 \frac{\partial}{\partial \gamma} + \xi_2 \frac{\partial}{\partial \lambda} + \eta_1 \frac{\partial}{\partial P} + \eta_2 \frac{\partial}{\partial Q} \tag{35}$$

Where infinitesimals are found to be

$$\xi_1 = 3a\gamma + b,\ \xi_2 = a\lambda + s(\gamma),\ \eta_1 = a(P-1),\ \eta_2 = -aQ + (P-1)s'(\alpha) \tag{36}$$

The boundary conditions restrict the infinitesimals (36) to the following form

$$\xi_1 = 3a\gamma + b,\ \xi_2 = a\lambda,\ \eta_1 = a(P-1),\ \eta_2 = -aQ \tag{37}$$

If we take a=1, b=0 and find the corresponding similarity variables and functions, we have

$$\zeta = \frac{\lambda}{\gamma^{1/3}},\ P = \gamma^{1/3}H(\zeta) + 1,\ Q = \gamma^{-1/3}L(\zeta),\ A = a_1\gamma^{1/3}$$
$$V = a_1\gamma^{-1/3},\ U = a_3\gamma^{1/3} + 1,\ f = a_4\gamma^{-1/3} \qquad (38)$$

Substituting the above variables into (22)-(24), we finally obtain the following ordinary differential system

$$H - \zeta H' + 3L' = 0 \qquad (39)$$

$$H^2 - \zeta HH' + 3LH' = 3\left[1 + (\kappa H')^2\right]^{\frac{n-1}{2}} H'' + 3\kappa De(n-1)\left[1 + (\kappa H')^2\right]^{\frac{n-3}{2}} (H')^2 H'' + a_4 \qquad (40)$$

$$H(0) = a_1,\ L(0) = \pm a_2,\ H(\infty) = a_3,\ a_4 = a_3^2 \qquad (41)$$

Therefore, in two different ways, we successfully reduced the partial differential system of three independent variables to ordinary differential equations. Equations can be solved numerically.

4. Numerical Solutions

In Figure 1 and 4, S and H functions, related to the x component of velocity, are plotted for different flow of behavior indexes. The boundary layers are qualitatively for n<1 (shear-thinning), n>1 (shear-thickening) and Newtonian for n=1 cases. Both the functions are observed to increase velocity with decreasing n. The influence of Deborah number is shown in Figure 2 and 5. The increase of Deborah number causes a rising of velocity, a fall in boundary layer thickness for shear-thinning fluids (n<1),but a reverse effect is observed for shear-thickeningfluids (n>1). A similar behavior for the κ parameter is observed for boundary layers. Effects of surface suction/injection on S, H are shown in Figure 3 and 6. Boundary layer thickness increases for injection, and decreases for suction cases.

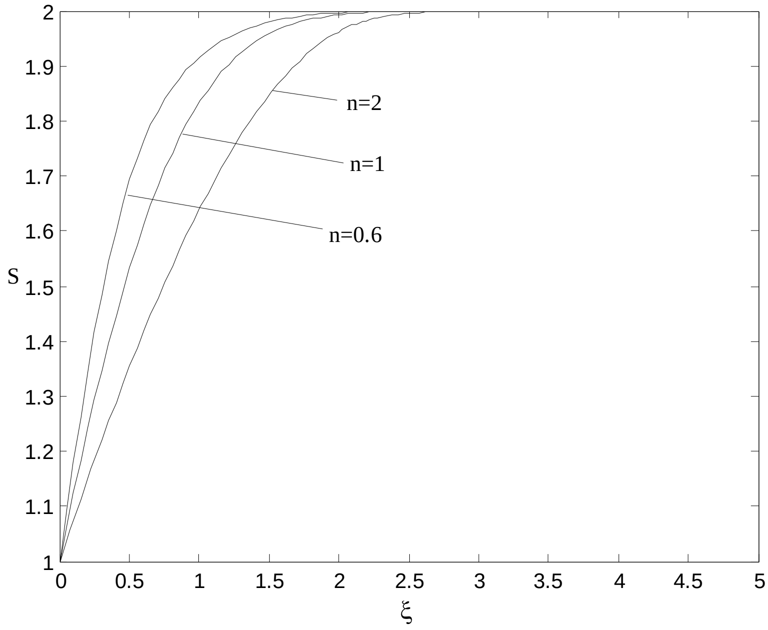

Figure1: Variation of S for different values of flow behaviour index (a_1=1, a_2=1, a_3=2, κ=3, De=5)

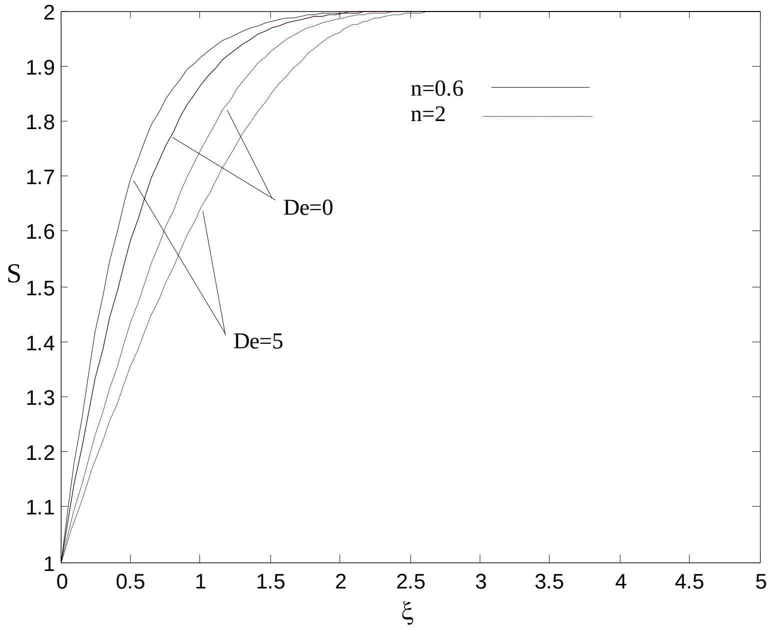

Figure2: Variation of S for different values of Deborah number

(a_1=1,a_2=1, a_3=2, κ=3)

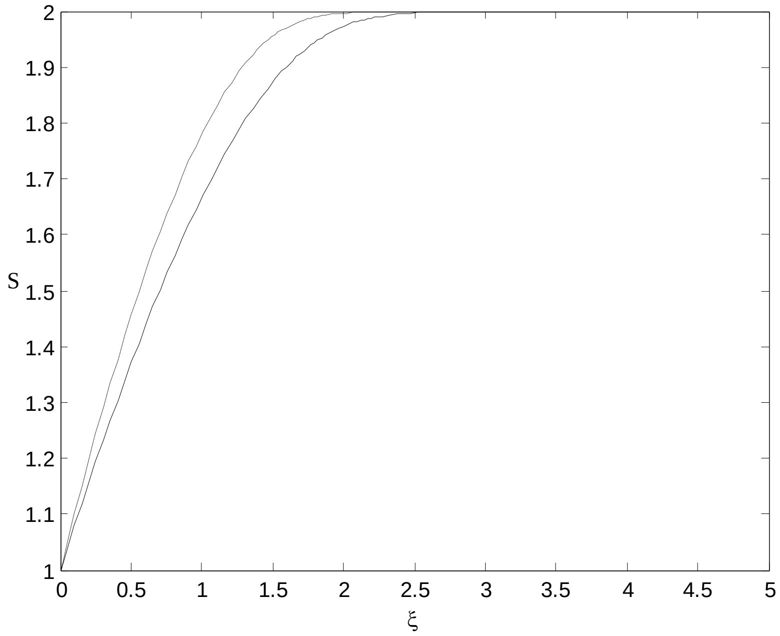

Figure 3: Variation of S for different value suction/injection parameter

(a_1=1, De=5, κ=2, n=2, a_3=2,   (a_2=1 (——), a_2=-1 (------))

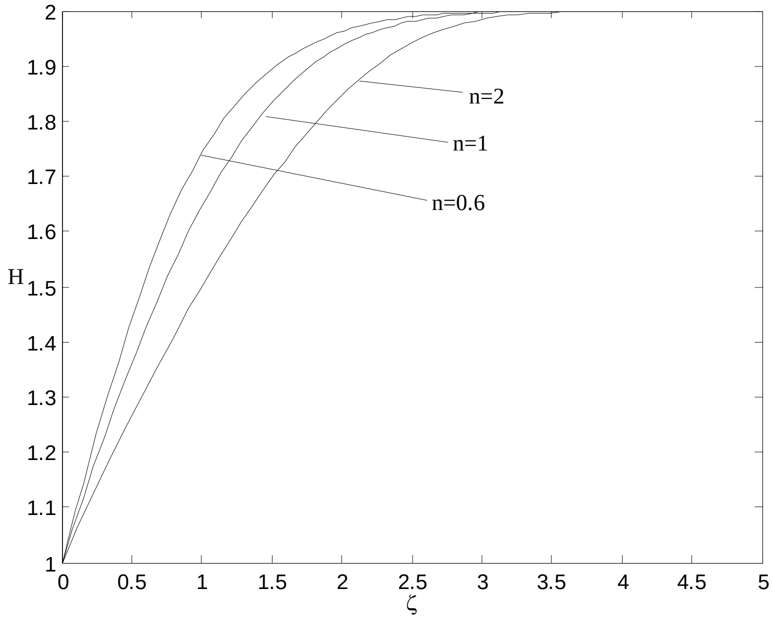

Figure 4: Variation of H for different values of flow behaviour index (a_1=1, a_2=1, a_3=2, κ=3, De=5)

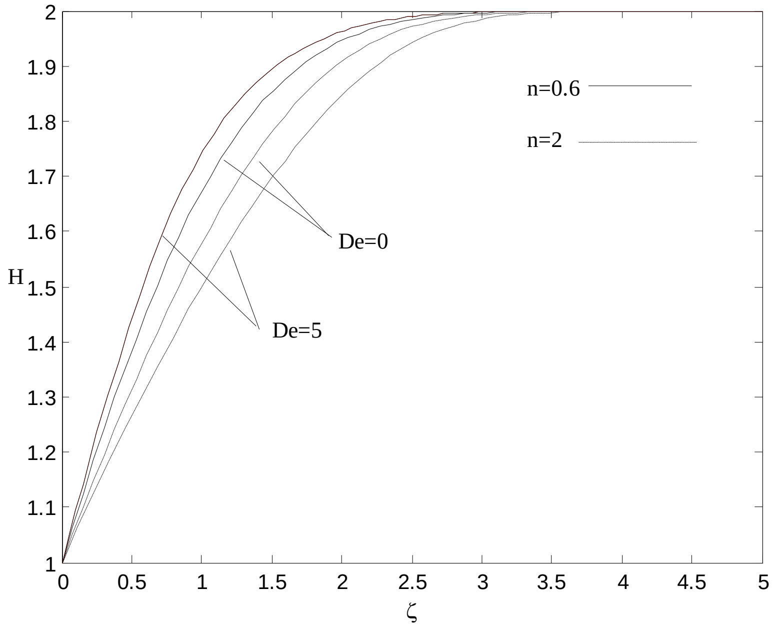

Figure 5: Variation of H for different values of Deborah number (a_1=1,a_2=1, a_3=2, κ=3)

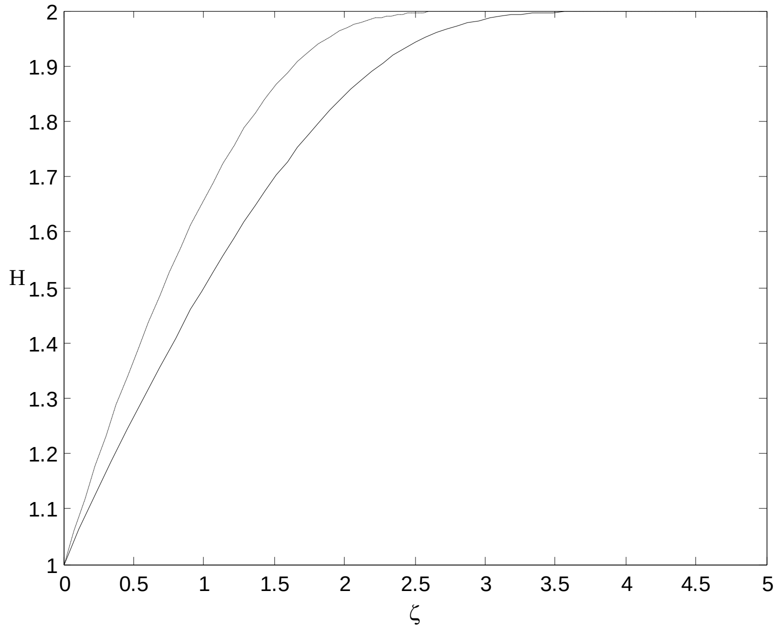

Figure 6: Variation of H for different values of suction/injection parameter

(a_1=1, De=5, κ=2, n=2, a_3=2, (a_2=1 (——), a_2=-1 (------))

5. Concluding Remarks

The classical unsteady boundary layer flow of a non-Newtonian Carreau fluid over a horizontal surface has been investigated in this paper. Lie Group theory is applied to the equations. The partial differential system is transformed to an ordinary differential system via two successive symmetry transformations. Boundary conditions are selected a moving surface, with suction or injection form. The resulting ordinary differential system is solved numerically. Effects of the flow behavior index, suction/injection parameter and Deborah number on the velocity profiles are analyzed. Velocity profiles are qualitatively different for flow behavior indexes less than 1 and greater than 1.Inertia boundary layer thickness increases for injection and decreases for suction. The increase of Deborah number causes a rising of velocity, a fall in boundary layer thickness for shear-thinning fluids (n<1), but a reverse effect is observed for shear-thickening fluids (n>1). A similar behavior for the κ parameter is observed.

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