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On Fractional Star Domination Numbers in Graphs

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Abstract:

This paper studies a digital problem of the discrete structure. In this paper, let $\gamma_{fs}'(G)$ be the fractional star domination number, we show the basic bounds of fractional star domination number in the general graph, and then determine the fractional star domination numbers for some special graphs, such as the generalized star graph of $S_n(m)$, the generalized wheel graph of $W_{m,n}$ and the generalized fan graph of $F_{m,n}$.

Keywords: Fractional star dominating function; Fractional star domination numbers; Generalized star graph; Generalized wheel graph; Generalized fan graph.

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1. Introduction

For the terminology and notations not defined here, we adopt those in Bondy and Murty [1] and Haynes, Hedetniemi and Slater [2] and consider simple graphs only. The theory of domination in graphs is an important content and an important part in graph theory. In recent years, the research content is more and more widely. The fractional domination in graphs [3] was first put forward and studied by G. S. Dmoke and S. T. Hedetniemi [3]. They give out some important research results and make a lot of researches. The reference [4] propose and study the signed edge domination in graphs, and then it occurs many concepts on the signed edge domination, such as the signed cycle domination [5], the signed star domination [6] and so on. The literature [7] put forward the concept of the fractional star domination, and based on it, we do further studies on the fractional star domination in graphs, which greatly improve and enrich the content of the theory of domination in graphs.

Let $G=(V,E)$ be a graph. For any vertex $v\in V(G)$, $E_G(v)=\{uv\in E \mid u\in V\}$ denotes the edge neighborhood of v in G. $E_G(v)$ is denoted by $E(v)$ simply. $N(v)$ denotes the vertex neighborhood of v in G. $d_G(v)=|N(v)|$ is called the degree of v in G, $\Delta=\Delta(G)$ and $\delta=\delta(G)$ denote the maximum degree and minimum degree of G respectively.

For convenience, $G=(V,E)$ is a graph, if $S\subseteq E(G)$, $f: E\to R$ is a real-valued function, we denote $f(S)=\sum_{e\in s} f(e)$.

In this paper, we introduce the concept of the Fractional star domination firstly.

Definition 1: Let $G=(V,E)$ be a connected graph. A real-valued function $f: E(G)\to[0,1]$ is said to be the Fractional star dominating function (FSDF) of G if $f(E(u))\geq 1$ holds for every vertex $u\in V(G)$. The Fractional star domination number $\gamma'_{fs}(G)$ of G is defined as $\gamma'_{fs}(G)=\min\{ f(E) \mid f$ is an FSDF of G}.

If f is such an FSDF that $\gamma'_{fs}(G)=f(E)$, then the function f is said to be a minimum FSDF.

Lemma 1 [7] Let G be a connected graph of order n, $\gamma'_{fs}(G)\geq \frac{n}{2}$, and this result is the best possible.

By the above definition of the fractional star dominating function, we can make a further research on it. In this paper, we use the existing boundaries, and then give the fractional star domination numbers of some special graphs.

2. Main Results

In this paper, we will introduce the definition of the generalized star graph firstly, it means that we put the rode of one endpoint which length is m together and the number of the rode is n. It is denoted by $S_n(m)$, and it is shown in the Figure 1.

Theorem 1: For the generalized star graph $S_n(m)$,

$$\gamma_{fs}'(S_n(m))=\begin{cases}\frac{mn}{2}+1, & if\ \ m\ is\ even;\\ \frac{n(m+1)}{2}, & if\ \ m\ is\ odd;\end{cases}.$$

Proof: Let $G=S_n(m)$, $V=V(G)=\{v\}\cup\{u_{i,j} \mid i\in\{1,2,\cdots,m\}, j\in\{1,2,\cdots,n\}\}$, such as the figure 1.

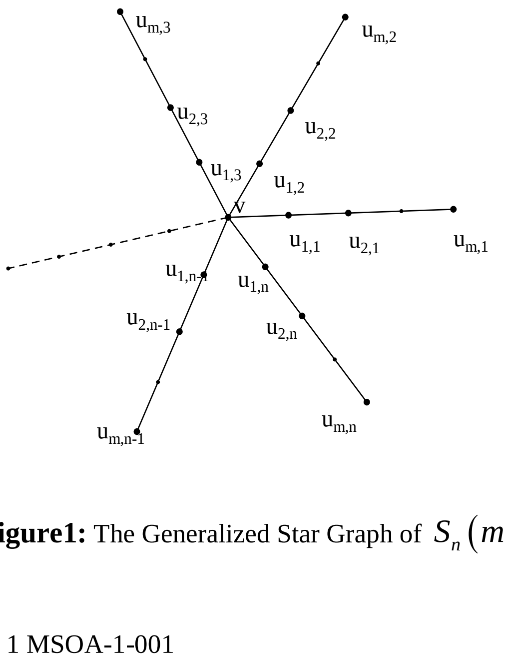

Figure1: The Generalized Star Graph of $S_n(m)$

Case 1. When m is even;

Let $E(G)=E_1\cup E_2\cup E_3$, and

$$E_1=\left\{vu_{1,j}\cup u_{i,j}u_{i+1,j}\mid i\in\{2,4,\cdots,m-2\},\ j\in\{1,2,\cdots,n\}\right\},$$

$$E_2=\left\{u_{i,j}u_{i+1,j}\mid i\in\{1,3,\cdots,m-3\},\ j\in\{1,2,\cdots,n\}\right\},$$

$$E_3=\left\{u_{m-1,j}u_{m,j}\mid j\in\{1,2,\cdots,n\}\right\}.$$

Define a function f of $E(S_n(m))$ as follows:

$$f(e)=\begin{cases}\dfrac{1}{n}, & if\ \ e\in E_1;\\ \dfrac{n-1}{n}, & if\ \ e\in E_2;\\ 1, & if\ \ e\in E_3;\end{cases}$$

Then $\forall u\in V$, $f(\mathrm{E(u)})\geq 1$. It is obvious to see that f is an FSDS of G. And hence

$$\gamma_{fs}'(G)\leq f(E)=\sum_{e\in E_1}f(e)+\sum_{e\in E_2}f(e)+\sum_{e\in E_3}f(e)$$

$$=\frac{m}{2}\cdot\frac{n}{n}+\left(\frac{m}{2}-1\right)\cdot\frac{n(n-1)}{n}+n$$

$$=\frac{mn}{2}+1.$$

By the definition 1, we know that for any vertex $u\in V(G)$, $f(E(u))\geq 1$, and also

$$\gamma_{fs}'(G)=\sum_{e\in E(G)}f(e)=f(E(v))+\sum_{j=1}^{n}\left(f(E(u_{2,j}))+f(E(u_{4,j}))+\cdots+f(E(u_{m,j}))\right)$$

$$\geq 1+\sum_{j=1}^{n}\left(\underbrace{1+1+\cdots+1}_{\frac{m}{2}}\right)=1+\frac{mn}{2}.$$

So when m is even, we have proved that $\gamma_{fs}'(S_n(m))=\dfrac{mn}{2}+1$.

Case 2. When m is odd;

Let $E(G)=E_1\cup E_2\cup E_3$, and

$$E_1=\left\{vu_{1,j}\cup u_{i,j}u_{i+1,j}\mid i\in\{2,4,\cdots,m-3\},\ j\in\{1,2,\cdots,n\}\right\},$$

$$E_2=\left\{u_{i,j}u_{i+1,j}\mid i\in\{1,3,\cdots,m-2\},\ j\in\{1,2,\cdots,n\}\right\},$$

$$E_3=\left\{u_{m-1,j}u_{m,j} \mid j\in\{1,2,\cdots,n\}\right\}.$$

Define a function f of $E\left(S_n\left(m\right)\right)$ as follows:

$$f\left(e\right)=\begin{cases}\frac{1}{n}, & if\ e\in E_1;\\ \frac{n-1}{n}, & if\ e\in E_2;\\ 1, & if\ e\in E_3;\end{cases}$$

Then $\forall u\in V$, $f\left(\mathrm{E(u)}\right)\geq 1$. It is obvious to see that f is an FSDS of G. And hence

$$\gamma_{fs}'\left(G\right)\leq f\left(E\right)=\sum_{e\in E_1}f\left(e\right)+\sum_{e\in E_2}f\left(e\right)+\sum_{e\in E_3}f\left(e\right)$$

$$=\frac{m-1}{2}\cdot\frac{n}{n}+\frac{m-1}{2}\cdot\frac{n\left(n-1\right)}{n}+n$$

$$=\frac{n\left(m+1\right)}{2}.$$

By the definition 1, we know that for any vertex $u\in V\left(G\right)$, $f\left(E\left(u\right)\right)\geq 1$, and also

$$\gamma_{fs}'\left(G\right)=\sum_{e\in E\left(G\right)}f\left(e\right)=\sum_{j=1}^{n}\left(f\left(E\left(u_{1,j}\right)\right)+f\left(E\left(u_{3,j}\right)\right)+\cdots+f\left(E\left(u_{m+1,j}\right)\right)\right)$$

$$\geq\sum_{j=1}^{n}\left(\underbrace{1+1+\cdots+1}_{\frac{m+1}{2}}\right)=\frac{n\left(m+1\right)}{2}.$$

So when m is odd, we have proved that $\gamma_{fs}'\left(S_n\left(m\right)\right)=\frac{n\left(m+1\right)}{2}$.

The proof of Theorem 1 is completed.

For the generalized star graph, we use one method to resolve its Fractional star domination number, and in the following tow theorem, we will use another method to do it.

Theorem 2: For the generalized wheel graph $W_{m,n}$, $\gamma_{fs}'\left(W_{m,n}\right)=\frac{mn+1}{2}$.

Proof: Let $G=W_{m,n}$, $V=V\left(G\right)=\{v\}\cup\left\{u_{i,j} \mid i\in\{1,2,\cdots,m\},\ j\in\{1,2,\cdots,n\}\right\}$, such as the figure 2.

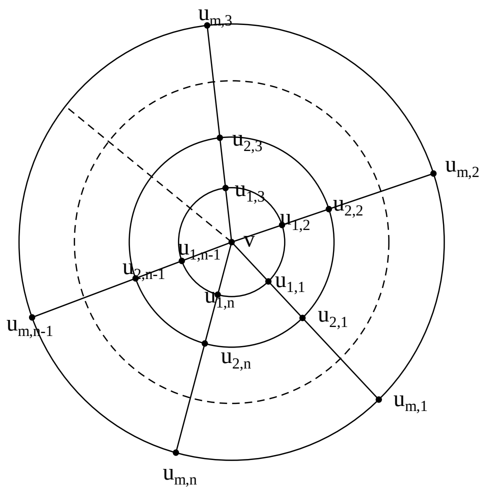

Figure 2: The Generalized Wheel Graph

On one hand, by the lemma 1, we know that $\gamma_{fs}'\left(W_{m,n}\right) \geq \frac{\left|V\left(W_{m,n}\right)\right|}{2} = \frac{mn+1}{2}$.

On the other hand, let $E_1 = \left\{vu_{1,j} \cup u_{i,j}u_{i+1,j} \mid i \in \{1,2,\cdots,m-1\}, j \in \{1,2,\cdots,n\}\right\}$,

$$E_2 = \left\{u_{i,j}u_{i,j+1} \mid i \in \{1,2,\cdots,m-1\}, j \in \{1,2,\cdots,n-1\}\right\},$$

$E_3 = \left\{u_{m,j}u_{m,j+1} \mid j \in \{1,2,\cdots,n-1\}\right\}$, and $E(G) = E_1 \cup E_2 \cup E_3$.

Define a function f of $E\left(W_{m,n}\right)$ as follows:

$$f(e) = \begin{cases} \frac{1}{n}, & if \ e \in E_1; \\ \frac{n-2}{2n}, & if \ e \in E_2; \\ \frac{n-1}{2n}, & if \ e \in E_3; \end{cases}$$

Then $\forall u \in V$, $f\left(\mathrm{E(u)}\right) \geq 1$. It is obvious to see that f is an FSDS of G .And hence

$$\gamma_{fs}'\left(W_{m,n}\right) \leq f(E) = \sum_{e\in E_1} f(e) + \sum_{e\in E_2} f(e) + \sum_{e\in E_3} f(e)$$

$$= \frac{n}{n} + (m-1)\frac{n(n-2)}{2n} + \frac{n(n-1)}{2n}$$

$$= \frac{mn+1}{2}.$$

In conclusion, $\gamma_{fs}'\left(W_{m,n}\right) = \frac{mn+1}{2}$.

Specially, when $m=1$, we have the following corollary.

Corollary 1 For the wheel graph W_n of size n, $\gamma_{fs}'\left(W_n\right) = \frac{n+1}{2}$.

Theorem 3: For the generalized fan graph $F_{m,n}$, $\gamma_{fs}'\left(F_{m,n}\right)=\frac{mn+1}{2}$.

Proof: Let $G=F_{m,n}$,$V=V\left(G\right)=\left\{v\right\}\cup\left\{u_{i,j}\mid i\in\left\{1,2,\cdots,m\right\},\ j\in\left\{1,2,\cdots,n\right\}\right\}$, such as the figure 3.

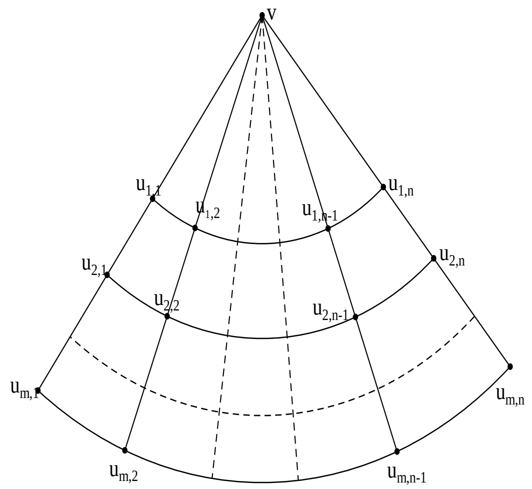

Figure 3: The Generalized Fan Graph

Case1. When n is odd;

(1) When n is odd and m is odd;

Let $E_1=\left\{vu_{1,1}\cup u_{i,n}u_{i+1,n}\mid i\in\left\{2,4,6,\cdots,m-1\right\}\right\}$, $E_2=\left\{u_{1,j}u_{1,j+1}\mid j\in\left\{2,4,6,\cdots,n-1\right\}\right\}$,

$E_3=\left\{u_{i,j}u_{i,j+1}\mid i\in\left\{2,3,\cdots,m\right\}\ j\in\left\{1,3,5,\cdots,n-2\right\}\right\}$.

Define a function f of $E\left(F_{m,n}\right)$ as follows:

$$f\left(e\right)=\begin{cases}1, & if\ \ e\in E_1\cup E_2\cup E_3;\\ 0, & if\ \ e\in E\setminus\left(E_1\cup E_2\cup E_3\right);\end{cases}$$

Then $\forall u\in V\left(G\right),\ f\left(\mathrm{E(u)}\right)\geq 1$. It is obvious to see that f is an FSDS of G . And hence

$$\gamma_{fs}'\left(F_{m,n}\right)\leq f\left(E\right)=\sum_{e\in E_1}f\left(e\right)+\sum_{e\in E_2}f\left(e\right)+\sum_{e\in E_3}f\left(e\right)$$

$$=\frac{m+1}{2}+\frac{n-1}{2}+\left(m-1\right)\cdot\frac{n-1}{2}$$

$$=\frac{mn+1}{2}.$$

And also by the lemma 1, we know that $\gamma_{fs}'\left(F_{m,n}\right)\geq\frac{\left|V\left(F_{m,n}\right)\right|}{2}=\frac{mn+1}{2}$.

So when n is odd and m is odd, we have proved that $\gamma_{fs}'\left(F_{m,n}\right)=\frac{mn+1}{2}$.

(2) When n is odd and m is even;

Let $E_1=\left\{vu_{1,1};vu_{1,2};vu_{1,n-1};vu_{1,n}\right\}$, $E_2=\left\{u_{1,1}u_{1,2};u_{1,n-1}u_{1,n}\right\}$,

$E_3=\left\{u_{i,j}u_{i+1,j}\mid i\in\left\{1,3,5,\cdots,m-1\right\}\ j\in\left\{3,4,5,\cdots,n-2\right\}\right\}$,

$$E_4=\left\{u_{i,j}u_{i,j+1} \mid i\in\{3,4,5,\cdots,m\}\ j\in\{1,n-1\}\right\}.$$

Define a function f of $E\left(F_{m,n}\right)$ as follows:

$$f(e)=\begin{cases}\frac{1}{4}, & if\ \ e\in E_1;\\ \frac{3}{4}, & if\ \ e\in E_2;\\ 1, & if\ \ e\in E_3\cup E_4;\\ 0,\ if\ \ e\in E\setminus\left(E_1\cup E_2\cup E_3\cup E_4\right);\end{cases}$$

Then $\forall u\in V(G)$, $f\left(\mathrm{E(u)}\right)\geq 1$. It is obvious to see that f is an FSDS of G .And hence

$$\gamma_{fs}{}'\left(F_{m,n}\right)\leq f(E)=\sum_{e\in E_1}f(e)+\sum_{e\in E_2}f(e)+\sum_{e\in E_3}f(e)+\sum_{e\in E_4}f(e)$$

$$=4\cdot\frac{1}{4}+2\cdot\frac{3}{4}+2(m-1)+(n-4)\cdot\frac{m}{2}$$

$$=\frac{mn+1}{2}.$$

And also by the lemma 1, we know that $\gamma_{fs}{}'\left(F_{m,n}\right)\geq\frac{\left|V\left(F_{m,n}\right)\right|}{2}=\frac{mn+1}{2}$.

So when n is odd and m is even, we have proved that $\gamma_{fs}{}'\left(F_{m,n}\right)=\frac{mn+1}{2}$.

In conclusion, $\gamma_{fs}{}'\left(F_{m,n}\right)=\frac{mn+1}{2}$.

Case 2: When n is even;

Let $E_1=\left\{vu_{1,j}\cup u_{i,j}u_{i+1,j} \mid i\in\{1,2,\cdots,m-1\},\ j\in\{1,2,\cdots,n\}\right\}$,

$$E_2=\left\{u_{i,j}u_{i,j+1} \mid i\in\{1,2,3,\cdots,m\},\ j\in\{1,3,5,\cdots,n-1\}\right\},$$

$$E_3=\left\{u_{m,j}u_{m,j+1} \mid j\in\{1,3,5,\cdots,n-1\}\right\}.$$

Define a function f of $E\left(F_{m,n}\right)$ as follows:

$$f(e)=\begin{cases}\frac{1}{n}, & if\ \ e\in E_1;\\ \frac{n-2}{n}, & if\ \ e\in E_2;\\ \frac{n-1}{n}, & if\ \ e\in E_3;\\ 0, & if\ \ e\in E\setminus\left(E_1\cup E_2\cup E_3\right);\end{cases}$$

Then $\forall u\in V(G)$, $f\left(\mathrm{E(u)}\right)\geq 1$. It is obvious to see that f is an FSDS of G .And hence

$$\gamma_{fs}{}'\left(F_{m,n}\right)\leq f(E)=\sum_{e\in E_1}f(e)+\sum_{e\in E_2}f(e)+\sum_{e\in E_3}f(e)$$

$$=m\cdot\frac{n}{n}+(m-1)\cdot\frac{n}{2}\cdot\frac{n-2}{n}+\frac{n}{2}\cdot\frac{n-1}{n}$$

$$=\frac{mn+1}{2}.$$

And also by the lemma 1, we know that $\gamma_{fs}'(F_{m,n})\geq\frac{|V(F_{m,n})|}{2}=\frac{mn+1}{2}$.

So when m is even, we have proved that $\gamma_{fs}'(F_{m,n})=\frac{mn+1}{2}$.

The proof of Theorem 3 is completed.

Specially, when $m=1$, we can get the fractional star domination number of the fan graph.

Corollary 2 For the fan graph F_n of size n, $\gamma_{fs}'(F_n)=\frac{n+1}{2}$.

3. Conclusions

The concept of the fractional star domination is growing so it seems natural to study the numbers of γ_{fs}'.We have given the values of the γ_{fs}' for some special graphs, it greatly improve and enrich the content of the theory of domination in graphs, and make the graph theory has more application value. But not all of the graphs exist the value of γ_{fs}', and we can do more study about this.

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